Supplemental appendix: Nonparametric inference on (conditional) quantile differences and interquantile ranges, using L-statistics

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Abstract

Appendix A contains fully detailed proofs; Appendix B, implementation details; Appendix C, detailed steps for CI construction; Appendix D, further approximation and intuition for the quantile difference CI calibration; Appendix E, additional simulations; and Appendix F, additional empirical analysis.

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A Mathematical proofs

We abbreviate Goldman and Kaplan (2017) as GK. For one-sample results, we omit the X subscript, e.g., writing $F(\cdot)$ instead of $F_X(\cdot)$.

First, we introduce notation. The following is one-sample notation; for two-sample notation, subscript x or y is added to indicate the sample. Vectors are always column vectors (unless otherwise noted) and in bold, e.g., $\mathbf{u} = (u_1, \ldots, u_J)'$.

Instead of (5), let the linearly interpolated fractional order statistics be

$$\hat{Q}_X^L\big(k/(n+1)\big) \equiv \hat{X}_{n:k}^L. \tag{24}$$

Let the idealized fractional order statistics be

$$\tilde{Q}_X^I\big(k/(n+1)\big) \equiv \tilde{X}_{n:k}^I, \quad \tilde{Q}_U^I\big(k/(n+1)\big) \equiv \tilde{U}_{n:k}^I. \tag{25}$$

Also, let

$$\tilde{Q}_X^I(\cdot) \equiv Q_X\big(\tilde{Q}_U^I(\cdot)\big). \tag{26}$$

Instead of (8), we use

$$u_j^h(\alpha) \equiv k_j^h(\alpha)/(n+1), \quad u_j^l(\alpha) \equiv k_j^l(\alpha)/(n+1). \tag{27}$$

From A1, $X_i \overset{iid}{\sim} F$, so $U_i \equiv F(X_i) \overset{iid}{\sim} \text{Unif}(0,1)$, with order statistics $U_{n:k}$. Let $\mathbf{u} = (u_1, \ldots, u_J)'$ be a generic vector with all $u_j \in (0,1)$; most commonly, we will have $\mathbf{u}$ equal $\boldsymbol{\tau}$, $\mathbf{u}_0^L$, $\mathbf{u}_0^H$, $\hat{\mathbf{u}}^L$, or $\hat{\mathbf{u}}^H$ (see definitions below). In any case, for convenience let $u_0 \equiv 0$ and $u_{J+1} \equiv 1$. Given $\mathbf{u}$, for all $j \in \{1, 2, \ldots, J\}$,

$$k_j \equiv \lfloor (n+1)u_j \rfloor, \qquad \epsilon_j \equiv (n+1)u_j - k_j,$$

where the $\epsilon_j \in [0,1)$ are interpolation weights as in (24). Let $\Delta\mathbf{k}$ denote the $(J+1)$-vector with elements $\Delta k_j = k_j - k_{j-1}$. Let $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_J)'$ be a fixed weight vector, and

$$\begin{aligned}
Y_j^{\mathbf{u}} &\equiv U_{n:k_j} \sim \text{Beta}(k_j, n+1-k_j), & \mathbf{Y}^{\mathbf{u}} &\equiv (Y_1^{\mathbf{u}}, \ldots, Y_J^{\mathbf{u}})', \\
\Delta\mathbf{Y}^{\mathbf{u}} &\equiv (Y_1, Y_2 - Y_1, \ldots, 1 - Y_J)' \sim \text{Dirichlet}(\Delta\mathbf{k}), \\
\Lambda_j^{\mathbf{u}} &\equiv U_{n:k_j+1} - U_{n:k_j} \sim \text{Beta}(1, n), & \boldsymbol{\Lambda}^{\mathbf{u}} &\equiv (\Lambda_1^{\mathbf{u}}, \ldots, \Lambda_J^{\mathbf{u}})', \\
\mathbb{W}^{\mathbf{u}} &\equiv \sqrt{n}\left(\sum_{j=1}^J \psi_j Q(Y_j^{\mathbf{u}}) - \sum_{j=1}^J \psi_j Q(u_j)\right), \\
\mathbb{W}_{\boldsymbol{\epsilon},\boldsymbol{\Lambda}}^{\mathbf{u}} &\equiv \mathbb{W}^{\mathbf{u}} + n^{1/2}\sum_{j=1}^J \epsilon_j\psi_j\Lambda_j^{\mathbf{u}}\big[Q'(u_j) + Q''(u_j)(Y_j^{\mathbf{u}} - u_j)\big], \\
\hat{\mathbf{u}}^H &\equiv \big\{u_j^H(\tilde{\alpha}(\hat{\boldsymbol{\gamma}}))\big\}_{j=1}^J, & \mathbf{u}_0^H &\equiv \big\{u_j^H(\tilde{\alpha}(\boldsymbol{\gamma}_0))\big\}_{j=1}^J,
\end{aligned} \tag{28}$$

where $u_j^H(\cdot)$ is defined in (34) and $Q(\cdot) = F^{-1}(\cdot)$ is the quantile function of interest, with first and second derivatives $Q'(\cdot)$ and $Q''(\cdot)$, and now $\tilde{\alpha}$ is explicitly written as a function of $\hat{\boldsymbol{\gamma}}$. We may also consider $\hat{\mathbf{u}}^L \equiv \big\{u_j^L\big(\tilde{\alpha}(\hat{\boldsymbol{\gamma}})\big)\big\}_{j=1}^J$ and $\mathbf{u}_0^L \equiv \big\{u_j^L\big(\tilde{\alpha}(\boldsymbol{\gamma}_0)\big)\big\}_{j=1}^J$, using notation from (34), but generally the "high" and "low" (H and L) endpoint results closely parallel each other, so we spend the most time on the H case. For random variables with a $\mathbf{u}$ superscript like $\mathbf{Y}^{\mathbf{u}}$ and $\mathbb{W}^{\mathbf{u}}$, if the vector $\mathbf{u}$ is clear from context, then the corresponding superscript may be omitted. For the sparsity (nuisance parameter) estimator, used only with $\mathbf{u} = \boldsymbol{\tau}$, with smoothing parameters m_j,

$$\begin{aligned} \Omega_j &\equiv U_{n:k_j+m_j} - U_{n:k_j-m_j} = \Omega_j^+ + \Omega_j^-, \\ \Omega_j^- &\equiv U_{n:k_j} - U_{n:k_j-m_j}, \quad \Omega_j^+ \equiv U_{n:k_j+m_j} - U_{n:k_j}. \end{aligned} \tag{29}$$

The values and distributions of the preceding variables are all understood to vary with n.

By construction, $\boldsymbol{\Delta}\mathbf{k}$ is a $(J+1)$-vector of natural numbers such that $\sum_{j=1}^{J+1} \Delta k_j = n+1$ and $k_j = \sum_{i=1}^{j} \Delta k_i$. In our applications to quantile inference, $\min_j\{\Delta k_j\} \to \infty$, and moreover all $\Delta k_j \asymp n$.

Define Condition $\star$ as satisfied by any value $\mathbf{y}$ if and only if

$$\max_j \bigl\{ n \Delta k_j^{-1/2} |\Delta y_j - \Delta k_j/n| \bigr\} \le 2\log(n). \qquad \text{Condition } \star$$

From GK Lemma 7(i), this implies the same $O\bigl(\log(n)\bigr)$ bound when centering at the mode or mean of ΔY_j:

$$\text{Condition } \star \implies \max_j \bigl\{ n \Delta k_j^{-1/2} |\Delta x_j - \Delta k_j/(n+1)| \bigr\} = O\bigl(\log(n)\bigr), \tag{30}$$

$$\text{Condition } \star \implies \max_j \bigl\{ n \Delta k_j^{-1/2} |\Delta x_j - (\Delta k_j - 1)/(n-J)| \bigr\} = O\bigl(\log(n)\bigr). \tag{31}$$

GK Lemma 7(iv,v) shows that Condition $\star$ is violated with $O(n^{-2})$ probability, which is negligible for all our results.

Essentially, Condition $\star$ plays the role of a cruder, weaker (but more broadly applicable in our case) law of iterated logarithm. For example, if $\Delta k_j/n = p_j$ is fixed and Δy_j were a sample average of iid random variables with mean p_j and unit variance, then

$$\limsup_{n\to\infty} \frac{n(\Delta y_j - p_j)}{\sqrt{n \log[\log(n)]}} = -\liminf_{n\to\infty} \frac{n(\Delta y_j - p_j)}{\sqrt{n \log[\log(n)]}} = \sqrt{2} \text{ a.s.},$$

so $|\Delta y_j - \Delta k_j/n| = O\bigl(n^{-1/2}\sqrt{\log[\log(n)]}\bigr)$ almost surely, compared to the weaker bound from Condition $\star$, $O\bigl(n^{-1/2}\log(n)\bigr)$ with probability $O(n^{-2})$. Actually, when $\Delta k_j \asymp n$, then Δy_j is equal in distribution to a uniform order statistic, which can be expressed as a sample average by the Bahadur representation (up to a smaller-order error), so the usual LIL could be used. However, we also need Condition $\star$ when $\Delta k_j = o(n)$ and a corresponding LIL is not available, so we use Condition $\star$ in all cases for simplicity since there is no detriment to the final results.

The following lemma is also useful, approximating the CI endpoint indices by standard normal quantiles.

Lemma 6. *Let $z_{1-\alpha}$ denote the $(1-\alpha)$-quantile of a standard normal distribution. From the definitions in (8) and (27), the values $u_j^l(\alpha)$ and $u_j^h(\alpha)$ can be approximated as*

$$\begin{aligned} u_j^l(\alpha) &= \tau_j - n^{-1/2} z_{1-\alpha} \sqrt{\tau_j(1-\tau_j)} - \frac{2\tau_j - 1}{6n}(z_{1-\alpha}^2 + 2) + O(n^{-3/2}), \\ u_j^h(\alpha) &= \tau_j + n^{-1/2} z_{1-\alpha} \sqrt{\tau_j(1-\tau_j)} - \frac{2\tau_j - 1}{6n}(z_{1-\alpha}^2 + 2) + O(n^{-3/2}). \end{aligned}$$

Proof. The proof is in GK. □

We also use the following result from Theorem 2(ii) of GK, reproduced here for convenience. With $L_0 \equiv \sum_{j=1}^{J} \psi_j Q_X(u_j)$, uniformly over $\mathbf{u} = \boldsymbol{\tau} + o(1)$,

$$\sup_{K\in\mathbb{R}} \left| \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \hat{X}^{L}_{n:(n+1)u_j} < L_0 + n^{-1/2}K \right) - \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \tilde{X}^{I}_{n:(n+1)u_j} < L_0 + n^{-1/2}K \right) \right| \\ = O(n^{-1}). \tag{32}$$

A.1 Proof of Theorem 1 (more general version)

Proof. As in the proof in GK, we assume that the realized values of all random variables satisfy Condition $\star$. By application of GK Lemma 7(iv,v) this induces at most $O(n^{-2})$ error in our calculations, which is asymptotically negligible.

We prove a more general version. Let

$$L_X^L \equiv \sum_{j=1}^{J} \psi_j \hat{X}^{L}_{n:(n+1)u_j}, \quad L_X^I \equiv \sum_{j=1}^{J} \psi_j \tilde{X}^{I}_{n:(n+1)u_j}. \tag{33}$$

Defining L_Y^L and L_Y^I similarly, and letting $L_0 = \sum_{j=1}^{J} \psi_j [Q_X(u_j) + Q_Y(u_j)]$, we need to show that uniformly over $\mathbf{u} = \boldsymbol{\tau} + o(1)$,

$$\sup_{K\in\mathbb{R}} \left| \mathrm{P}\big(L_X^L + L_Y^L < L_0 + n^{-1/2}K \big) - \mathrm{P}\big(L_X^I + L_Y^I < L_0 + n^{-1/2}K \big) \right| = O(n^{-1}).$$

This follows because for any probability distribution $G(\cdot)$ and function $h(\cdot)$,

$$\left| \int h(x)\, dG(x) \right| \le \sup_x |h(x)|,$$

so, using $L_X^L \perp\!\!\!\perp L_Y^L$ from A1,

$$\begin{aligned}
&\mathrm{P}\big(L_X^L + L_Y^L < K\big) \\
&\quad = \int_{\mathbb{R}} \mathrm{P}\big(L_Y^L < K - x\big)\, dF_{L_X^L}(x) \\
&\quad = \int_{\mathbb{R}} \mathrm{P}\big(L_Y^I < K - x\big)\, dF_{L_X^L}(x) + \overbrace{\int_{\mathbb{R}} \overbrace{\big[\mathrm{P}\big(L_Y^L < K - x\big) - \mathrm{P}\big(L_Y^I < K - x\big)\big]}^{\text{uniformly } O(n^{-1}) \text{ by (32)}}\, dF_{L_X^L}(x)}^{\text{uniformly } O(n^{-1})} \\
&\quad = \mathrm{P}\big(L_Y^I + L_X^L < K\big) + O\big(n^{-1}\big) \\
&\quad = \int_{\mathbb{R}} \mathrm{P}\big(L_X^I < K - x\big)\, dF_{L_Y^I}(x) \\
&\qquad + \overbrace{\int_{\mathbb{R}} \overbrace{\big[\mathrm{P}\big(L_X^L < K - x\big) - \mathrm{P}\big(L_X^I < K - x\big)\big]}^{\text{uniformly } O(n^{-1}) \text{ by (32)}}\, dF_{L_Y^I}(x)}^{\text{uniformly } O(n^{-1})} + O(n^{-1}) \\
&\quad = \mathrm{P}\big(L_X^I + L_Y^I < K\big) + O(n^{-1})
\end{aligned}$$

uniformly over $K \in \mathbb{R}$ and $\mathbf{u} = \boldsymbol{\tau} + o(1)$. $\square$

A.2 Proof of Theorem 2

Proof. Applying (32) to get the first equality below, actual CP is

$$
\begin{aligned}
&\mathrm{P}\left(\left\{\bigcap_{j=1}^{J}\left\{\hat{Q}_X^L\left(u_j^h(\tilde{\alpha}/2)\right) > Q(\tau_j)\right\}\right\} \bigcap \left\{\bigcap_{j=1}^{J}\left\{\hat{Q}_X^L\left(u_j^l(\tilde{\alpha}/2)\right) < Q(\tau_j)\right\}\right\}\right) \\
&= \overbrace{\mathrm{P}\left(\left\{\bigcap_{j=1}^{J}\left\{\tilde{Q}_X^I\left(u_j^h(\tilde{\alpha}/2)\right) > Q(\tau_j)\right\}\right\} \bigcap \left\{\bigcap_{j=1}^{J}\left\{\tilde{Q}_X^I\left(u_j^l(\tilde{\alpha}/2)\right) < Q(\tau_j)\right\}\right\}\right)}^{\text{use (26)}} + O(n^{-1}) \\
&= \overbrace{\mathrm{P}\left(\left\{\bigcap_{j=1}^{J}\left\{\tilde{Q}_U^I\left(u_j^h(\tilde{\alpha}/2)\right) > \tau_j\right\}\right\} \bigcap \left\{\bigcap_{j=1}^{J}\left\{\tilde{Q}_U^I\left(u_j^l(\tilde{\alpha}/2)\right) < \tau_j\right\}\right\}\right)}^{=1-\alpha \text{ by (10)}} + O(n^{-1}) \\
&= 1 - \alpha + O(n^{-1}).
\end{aligned}
$$

The application of (32) above follows from the Cramér–Wold device. $\square$

A.3 Proof of Theorem 3

We first state (and then prove) Theorem 7, which is a more general version of Theorem 3. The X subscript is dropped for notational simplicity. Let $Q(\boldsymbol{\tau}) \equiv \big(Q(\tau_1), \ldots, Q(\tau_J)\big)'$, and similarly for $Q'(\boldsymbol{\tau})$, $\widehat{Q'}(\boldsymbol{\tau})$, etc.

For quantile index vector $\boldsymbol{\tau} \in (0,1)^J$ and weights $\boldsymbol{\psi} \in \mathbb{R}^J$, we construct a CI for $D = \sum_{j=1}^{J} \psi_j Q(\tau_j)$. The theorem stated in the main text is for the special case with $\boldsymbol{\psi} = (-1, 1)'$ and $\boldsymbol{\tau} = (0.25, 0.75)'$.

Using (27), let

$$
\begin{aligned}
u_j^H(\alpha) &\equiv \mathbb{1}\{\psi_j > 0\} u_j^h(\alpha) + \mathbb{1}\{\psi_j < 0\} u_j^l(\alpha), \\
u_j^L(\alpha) &\equiv \mathbb{1}\{\psi_j > 0\} u_j^l(\alpha) + \mathbb{1}\{\psi_j < 0\} u_j^h(\alpha).
\end{aligned}
\tag{34}
$$

In the notation of (34), the lower one-sided CI for D is

$$
\left(-\infty, \sum_{j=1}^{J} \psi_j \hat{Q}_X^L\big(u_j^H(\tilde{\alpha})\big)\right),
\tag{35}
$$

where $\tilde{\alpha}$ implicitly depends on the $\widehat{Q'}(\tau_j)$ and satisfies

$$
1 - \alpha = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \widehat{Q'}(\tau_j) \Big[\tilde{Q}_U^I\big(u_j^H(\tilde{\alpha})\big) - \tau_j\Big] > 0\right).
\tag{36}
$$

For an upper one-sided CI, the analogs of (35) and (36) are

$$\left(\sum_{j=1}^{J} \psi_j \hat{Q}_X^L\big(u_j^L(\tilde{\alpha})\big), \infty\right), \quad 1-\alpha = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \widehat{Q'}(\tau_j)\Big[\tilde{Q}_U^I\big(u_j^L(\tilde{\alpha})\big) - \tau_j\Big] < 0\right). \tag{37}$$

Theorem 7. *Let A1 and A2 hold.*

(i) The one-sided lower and upper CIs in (35) *and* (37) *have CPE of order* $O\big(n^{-1/2}\log(n)\big)$ *if all* $\widehat{Q'}(\tau_j)$ *are estimated by* (12) *with smoothing parameters* m_j *having rate larger than* $n^{1/2}$ *and smaller than* $n^{3/4}$.

(ii) The two-sided CI formed by the intersection of upper and lower one-sided $1-\alpha/2$ *CIs has CPE of order* $O\big(n^{-2/3}\log(n)\big)$ *if all* $\widehat{Q'}(\tau_j)$ *are estimated by* (12) *with* $m_j \asymp n^{2/3}$.

(iii) The asymptotic probabilities of excluding $D_n = \boldsymbol{\psi}'\big[Q(\boldsymbol{\tau}) + \boldsymbol{\kappa} n^{-1/2}\big]$ *from lower one-sided (l), upper one-sided (u), and equal-tailed two-sided (t) CIs (i.e., asymptotic power of the corresponding hypothesis tests) are*

$$\mathcal{P}_n^l(D_n) \to \Phi(z_\alpha + S),\ \ \mathcal{P}_n^u(D_n) \to \Phi(z_\alpha - S),\ \ \mathcal{P}_n^t(D_n) \to \Phi\big(z_{\alpha/2} + S\big) + \Phi\big(z_{\alpha/2} - S\big),$$

where $S \equiv \boldsymbol{\psi}'\boldsymbol{\kappa}/\sqrt{\mathcal{V}_{\boldsymbol{\psi}}}$ *and*

$$\mathcal{V}_{\boldsymbol{\psi}} \equiv \sum_{i=1}^{J}\sum_{j=1}^{J} \psi_i\psi_j \frac{\min\{\tau_i,\tau_j\} - \tau_i\tau_j}{f(Q(\tau_i))f(Q(\tau_j))}. \tag{38}$$

Proof. We focus on the lower one-sided CI first; the upper one-sided results are entirely parallel.

To be more explicit about the dependence of $\tilde{\alpha}$ on the nuisance parameter, for a general value $\mathbf{g} = (g_1, \ldots, g_J)'$, let $\tilde{\alpha}[\mathbf{g}]$ satisfy

$$1-\alpha = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j g_j \Big[\tilde{Q}_U^I\big(u_j^H(\tilde{\alpha}[\mathbf{g}])\big) - \tau_j\Big] > 0\right). \tag{39}$$

This is (36) with g_j replacing $\widehat{Q'}(\tau_j)$. Correspondingly, we write $\hat{\mathbf{u}}$ for the vector of quantile indices selected to form CI endpoints given estimates

$$\hat{\boldsymbol{\gamma}} \equiv \widehat{Q'}(\boldsymbol{\tau}) \equiv \big(\widehat{Q'}(\tau_1), \ldots, \widehat{Q'}(\tau_J)\big)', \tag{40}$$

and $\mathbf{u}_0^H$ is the vector of quantile indices that would be selected if the true

$$\boldsymbol{\gamma} \equiv Q'(\boldsymbol{\tau}) \tag{41}$$

were known. Note that (39) is invariant to scaling $\mathbf{g}$ by a constant scalar, so we could also divide by the first element in the vector to normalize the first element to be one, e.g., $\boldsymbol{\gamma} = (1, Q'(\tau_2)/Q'(\tau_1), \ldots, Q'(\tau_J)/Q'(\tau_1))'$, which will be used later. For the lower one-sided case,

$$\hat{\mathbf{u}}^H \equiv \big\{u_j^H(\tilde{\alpha}[\hat{\boldsymbol{\gamma}}])\big\}_{j=1}^J \text{ and } \mathbf{u}_0^H \equiv \big\{u_j^H(\tilde{\alpha}[\boldsymbol{\gamma}])\big\}_{j=1}^J.$$

The CP of the lower one-sided CI can be decomposed into different components. We use notation from (28). We also use an implication of GK Lemma 8(i). Translating to our present context, that lemma states

$$\left|\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U(\hat{\mathbf{u}}^H)\right) - Q(\hat{\mathbf{u}}^H)\right] - \mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}}\right| = O\left(n^{-3/2}[\log(n)]^3\right),$$

so for any $t \in \mathbb{R}$,

$$\begin{aligned}
&\mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U(\hat{\mathbf{u}}^H)\right) - Q(\hat{\mathbf{u}}^H)\right] > t \text{ and } \mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} < t\right) \\
&= \overbrace{\mathrm{P}\left(t - O\left(n^{-3/2}[\log(n)]^3\right) < \mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} < t\right)}^{\text{use MVT}} \\
&\le O\left(n^{-3/2}[\log(n)]^3\right) \overbrace{\sup_{w \in [t - O\left(n^{-3/2}[\log(n)]^3\right), t]} f_{\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}}}(w)}^{=O(1) \text{ by GK Lemma 8(ii)}} \\
&= O\left(n^{-3/2}[\log(n)]^3\right),
\end{aligned}$$

and switching the $<$ and $>$ leaves the rate unchanged. Thus,

$$\begin{aligned}
&\left|\mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U(\hat{\mathbf{u}}^H)\right) - Q(\hat{\mathbf{u}}^H)\right] > t\right) - \mathrm{P}\left(\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} > t\right)\right| \\
&\le \overbrace{\left|\mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U(\hat{\mathbf{u}}^H)\right) - Q(\hat{\mathbf{u}}^H)\right] > t \text{ and } \mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} < t\right)\right|}^{=O\left(n^{-3/2}[\log(n)]^3\right)} \\
&\quad + \overbrace{\left|\mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U(\hat{\mathbf{u}}^H)\right) - Q(\hat{\mathbf{u}}^H)\right] < t \text{ and } \mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} > t\right)\right|}^{=O\left(n^{-3/2}[\log(n)]^3\right)} \\
&= O\left(n^{-3/2}[\log(n)]^3\right).
\end{aligned} \tag{42}$$

The lower one-sided CP is

$$\begin{aligned}
&\mathrm{P}\left(\boldsymbol{\psi}'\hat{Q}^L_X\left(\hat{\mathbf{u}}^H\right) > \boldsymbol{\psi}'Q(\boldsymbol{\tau})\right) \\
&= \overbrace{\mathrm{P}\left(\boldsymbol{\psi}'\tilde{Q}^I_X\left(\hat{\mathbf{u}}^H\right) > \boldsymbol{\psi}'Q(\boldsymbol{\tau})\right) + O(n^{-1})}^{\text{by (32)}} \\
&= \mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U\left(\hat{\mathbf{u}}^H\right)\right) - Q(\hat{\mathbf{u}}^H)\right] > \sqrt{n}\boldsymbol{\psi}'\left[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)\right]\right) + O(n^{-1}) \\
&= \mathrm{P}\left(\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} > \sqrt{n}\boldsymbol{\psi}'\left[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)\right]\right) \\
&\quad + \left[\mathrm{P}\left(\sqrt{n}\boldsymbol{\psi}'\left[Q\left(\tilde{Q}^I_U\left(\hat{\mathbf{u}}^H\right)\right) - Q(\hat{\mathbf{u}}^H)\right] > \sqrt{n}\boldsymbol{\psi}'\left[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)\right]\right)\right. \\
&\qquad \left. - \mathrm{P}\left(\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} > \sqrt{n}\boldsymbol{\psi}'\left[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)\right]\right)\right] \\
&\quad + O(n^{-1}) \\
&= \mathrm{P}(\mathbb{W}^{\mathbf{u}_0^H}_{\mathbf{C},\boldsymbol{\Lambda}} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)]) \\
&\quad + \overbrace{\mathrm{P}\left(\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}} > \sqrt{n}\boldsymbol{\psi}'\left[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)\right]\right) - \mathrm{P}(\mathbb{W}^{\mathbf{u}_0^H}_{\mathbf{C},\boldsymbol{\Lambda}} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)])}^{E_h}
\end{aligned}$$

$$
+\overbrace{\left[O\Big(n^{-3/2}[\log(n)]^3\Big)\right]}^{\text{by (42)}} + O(n^{-1})
$$

$$
= \overbrace{\mathrm{P}\left(\sum_{j=1}^{J}\psi_j\gamma_j\Big(\tilde{Q}_U^I(u_{0,j}^H)-\tau_j\Big)>0\right)}^{=1-\alpha \text{ by (39)}} + T_h + E_h + O(n^{-1}), \tag{43}
$$

where

$$
\begin{aligned}
T_h &= \mathrm{P}\Big(\mathbb{W}_{\mathbf{C},\mathbf{\Lambda}}^{\mathbf{u}_0^H} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\mathbf{u}_0^H)]\Big) - \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\gamma_j\Big(\tilde{Q}_U^I(u_{0,j}^H)-\tau_j\Big)>0\right) \\
&= \overbrace{\mathrm{P}\Big(\sqrt{n}\boldsymbol{\psi}'\Big[Q\Big(\tilde{Q}_U^I(\mathbf{u}_0^H)\Big)-Q(\mathbf{u}_0^H)\Big] > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\mathbf{u}_0^H)]\Big) + O\Big(n^{-3/2}[\log(n)]^3\Big)}^{\text{by (42)}} \\
&\quad - \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\gamma_j\Big(\tilde{Q}_U^I(u_{0,j}^H)-\tau_j\Big)>0\right) \\
&= \mathrm{P}\Big(\sqrt{n}\boldsymbol{\psi}'\Big[Q\Big(\tilde{Q}_U^I(\mathbf{u}_0^H)\Big)-Q(\boldsymbol{\tau})\Big]>0\Big) - \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\gamma_j\Big(\tilde{Q}_U^I(u_{0,j}^H)-\tau_j\Big)>0\right) \\
&\quad + O\Big(n^{-3/2}[\log(n)]^3\Big) \\
&= \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big[Q\Big(\tilde{Q}_U^I(u_{0,j}^H)\Big)-Q(\tau_j)\Big]>0\right) - \mathrm{P}\left(\sum_{j=1}^{J}\psi_j Q'(\tau_j)\Big(\tilde{Q}_U^I(u_{0,j}^H)-\tau_j\Big)>0\right) \\
&\quad + O\Big(n^{-3/2}[\log(n)]^3\Big),
\end{aligned} \tag{44}
$$

$$
E_h = \mathbb{E}\Big\{\mathrm{P}(\mathbb{W}_{\mathbf{C},\mathbf{\Lambda}}^{\hat{\mathbf{u}}^H} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\hat{\mathbf{u}}^H)] \mid \hat{\boldsymbol{\gamma}}) - \mathrm{P}(\mathbb{W}_{\mathbf{C},\mathbf{\Lambda}}^{\mathbf{u}_0^H} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}})\Big\}. \tag{45}
$$

The term T_h captures the error in the first-order Taylor approximation of $Q\big(\tilde{Q}_U^I(u_{0,j}^H)\big)-Q(\tau_j)$, and E_h captures estimation error in $\hat{\boldsymbol{\gamma}}$. The upper one-sided derivation yields similar terms, denoted T_l and E_l.

The proof of part (i) follows by applying Lemmas 8 and 9, which respectively have $T_h = O\big(n^{-1/2}\log(n)\big)$ and $E_h = O\big(m^{-1}\log(n)+(m/n)^2\big)$ for common smoothing parameter rate m (so $m_j \asymp m$ for all j), and similarly for T_l and E_l, which correspond to the upper one-sided CI. Plugging these into (43) gives one-sided CPE equal to $O\big(n^{-1/2}\log(n)\big)+O\big(m^{-1}\log(n)+(m/n)^2\big)$. As long as $n^{1/2}\lesssim m \lesssim n^{3/4}$, the dominant CPE term is order $O\big(n^{-1/2}\log(n)\big)$.

The proof of part (ii) also follows by applying Lemmas 8 and 9, which also give $T_h+T_l = O\big(n^{-1}[\log(n)]^2\big)$. Thus, CPE is $O\big(n^{-1}[\log(n)]^2\big)+O\big(m^{-1}\log(n)+(m/n)^2\big)$. Now, the second term dominates, and it is minimized by $m\asymp n^{2/3}$, leaving CPE of order $O\big(n^{-2/3}\log(n)\big)$.

The proof of part (iii) remains. One-sided power against $H_0: D_n = \boldsymbol{\psi}'[Q(\boldsymbol{\tau})+\boldsymbol{\kappa}n^{-1/2}]$ with $\boldsymbol{\psi}'\boldsymbol{\kappa}>0$ is the probability that D_n is not contained in the lower one-sided CI. Below, $\tilde{u}_j$ comes from the mean value theorem and lies between τ_j and u_j^H. Since $u_j^H\to\tau_j$ by Lemma 6, $\tilde{u}_j\to\tau_j$, so for large enough n, all $\tilde{u}_j$ lie within an arbitrarily small neighborhood of τ_j and thus A2 uniformly bounds $Q''(\tilde{u}_j) = O(1)$. The CI exclusion probability is

$$\begin{aligned}
\mathcal{P}_n^l(D_n) &= \mathrm{P}\left\{\sum_{j=1}^{J} \psi_j \left[\hat{Q}_X^L(u_j^H(\tilde{\alpha}_j)) - Q(\tau_j)\right] < n^{-1/2}\boldsymbol{\psi}'\boldsymbol{\kappa}\right\} \\
&= \mathrm{P}\left\{\boldsymbol{\psi}'\hat{Q}_X^L\big(\mathbf{u}^H(\tilde{\boldsymbol{\alpha}})\big) - \boldsymbol{\psi}'Q\big(\mathbf{u}^H(\tilde{\boldsymbol{\alpha}})\big) < n^{-1/2}\boldsymbol{\psi}'\boldsymbol{\kappa} - \boldsymbol{\psi}'\big[Q\big(\mathbf{u}^H(\tilde{\boldsymbol{\alpha}})\big) - Q(\boldsymbol{\tau})\big]\right\} \\
&= \mathrm{P}\Bigg\{\sqrt{n}\boldsymbol{\psi}'\left[\hat{Q}_X^L\big(\mathbf{u}^H(\tilde{\boldsymbol{\alpha}})\big) - Q\big(\mathbf{u}^H(\tilde{\boldsymbol{\alpha}})\big)\right] \\
&\qquad < \boldsymbol{\psi}'\boldsymbol{\kappa} - \sqrt{n}\sum_{j=1}^{J}\psi_j\Bigg[Q'(\tau_j)(u_j^H - \tau_j) + (1/2)\overbrace{Q''(\tilde{u}_j)}^{=O(1)}\overbrace{(u_j^H - \tau_j)^2}^{=O(n^{-1})\text{ by Lemma 6}}\Bigg]\Bigg\} \\
&= \overbrace{\Phi\left(\frac{\sum_{j=1}^{J}\psi_j\kappa_j - \psi_j Q'(\tau_j)\overbrace{\sqrt{n}\big[u_j^H(\tilde{\alpha}_j) - \tau_j\big]}^{\text{apply Lemma 6}} + O(n^{-1/2})}{\sqrt{\hat{\mathcal{V}}_{\boldsymbol{\psi}}}}\right) + O\Big(n^{-1/2}[\log(n)]^3\Big)}^{\text{by GK Lemma 8}} \\
&= \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi}}}} - \overbrace{\frac{1}{\sqrt{\mathcal{V}_{\boldsymbol{\psi}}}}\sum_{j=1}^{J}\psi_j Q'(\tau_j) z_{1-\tilde{\alpha}_j}\sqrt{\tau_j(1-\tau_j)}}^{\to z_{1-\alpha}\text{ to control size when }\boldsymbol{\kappa}=\mathbf{0}} + O(n^{-1/2})\right) + O\Big(n^{-1/2}[\log(n)]^3\Big) \\
&\to \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi}}}} - z_{1-\alpha}\right) = \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi}}}} + z_{\alpha}\right),
\end{aligned}$$

where

$$\hat{\mathcal{V}}_{\boldsymbol{\psi}} \equiv \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j\frac{\min\Big\{u_i^H(\tilde{\alpha}_i), u_j^H(\tilde{\alpha}_j)\Big\} - u_i^H(\tilde{\alpha}_i)u_j^H(\tilde{\alpha}_j)}{f\big(Q(u_i^H(\tilde{\alpha}_i))\big)f\Big(Q(u_j^H(\tilde{\alpha}_j))\Big)} \to \mathcal{V}_{\boldsymbol{\psi}}.$$

These results are invariant to choosing a single $\tilde{\alpha}$ or different $\tilde{\alpha}_j$ because the term involving the $\tilde{\alpha}_j$ must equal z_α in order to control size. In the special case $\psi = 1$ for a single quantile, then $\tilde{\alpha} = \alpha$, and the result reduces to the result in GK Theorem 4.

The upper one-sided case follows similarly.

For the two-sided case, since the two-sided CI is the intersection of the upper and lower one-sided $1 - \alpha/2$ CIs, the exclusion probability is

$$\begin{aligned}
\mathcal{P}_n^t(D_n) &= \mathrm{P}\left(D_n \notin \left[\sum_{j=1}^{J}\psi_j\hat{Q}_X^L\big(u_j^L(\tilde{\alpha}/2)\big), \sum_{j=1}^{J}\psi_j\hat{Q}_X^L\big(u_j^H(\tilde{\alpha}/2)\big)\right]\right) \\
&= \overbrace{\mathrm{P}\left(\sum_{j=1}^{J}\psi_j\hat{Q}_X^L\big(u_j^H(\tilde{\alpha}/2)\big) < D_n\right)}^{\mathcal{P}_n^l(D_n)\text{ with }\alpha/2} + \overbrace{\mathrm{P}\left(\sum_{j=1}^{J}\psi_j\hat{Q}_X^L\big(u_j^L(\tilde{\alpha}/2)\big) > D_n\right)}^{\mathcal{P}_n^l(D_n)\text{ with }\alpha/2}
\end{aligned}$$

$$\to \Phi\left(z_{\alpha/2} + \frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\psi}}}\right) + \Phi\left(z_{\alpha/2} - \frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\psi}}}\right). \qquad \square$$

A.3.1 CPE from Taylor Approximations: T_h, T_l

Lemma 8. *Under the assumptions of Theorem 7, the term T_h from (43) is of order $O\big(n^{-1/2}\log(n)\big)$, and similarly $T_l = O\big(n^{-1/2}\log(n)\big)$ for the corresponding upper one-sided term. Additionally, $T_h + T_l = O\big(n^{-1}[\log(n)]^2\big)$.*

Proof. For this proof, we introduce the scaled and centered

$$\Delta_j^H \equiv \sqrt{n}\Big(\tilde{Q}_U^I\big[u_j^H(\tilde{\alpha})\big] - u_j^H(\tilde{\alpha})\Big), \quad \Delta_j^L \equiv \sqrt{n}\Big(\tilde{Q}_U^I\big[u_j^L(\tilde{\alpha})\big] - u_j^L(\tilde{\alpha})\Big). \tag{46}$$

We split the overall probability into two pieces: one where

$$\big|\Delta_j^H\big| \le 2\sqrt{\log(n)} \text{ and } |\Delta_j^L| \le 2\sqrt{\log(n)}, \quad j = 1, \dots, J, \tag{47}$$

and one for the rest. Note that (47) is just a slight variant of Condition $\star$. The probability of (47) being violated is bounded with the help of GK Lemma 7(iv): applied here, for any $\tilde{Q} \sim \text{Beta}\big(u(n+1), (1-u)(n+1)\big)$, $\text{P}\big(\sqrt{n}|\tilde{Q} - u| > a_n\big) = O\big(a_n^{-1}\exp\{-a_n^2/2\}\big)$. In terms of $\Delta = \sqrt{n}(\tilde{Q} - u)$, letting $a_n = \sqrt{2\log(n)}$,

$$\text{P}\big(|\Delta| > 2\sqrt{\log(n)}\big) = O\big(\exp\{-(1/2)[2\sqrt{\log(n)}]^2\}\big) = O\big(\exp\{\log(n)(-2)\}\big) = O(n^{-2}). \tag{48}$$

So, with $a_n^2 = 2\log(n)$, since J is fixed, the probability that any $\Delta_j^2 > a_n^2$ (i.e., the probability of the union of such events) is of the same order of magnitude by Boole's Inequality. This order of magnitude is much smaller than that of the dominant terms in T_h and T_l stated in the theorem. We can now focus on the case where all the Δ_j are less than $\sqrt{2\log(n)}$ in absolute value.

Let $\phi_{\underline{\mathbf{V}}}(\cdot)$ denote the PDF of a multivariate normal distribution with mean zero and covariance matrix $\underline{\mathbf{V}}$. From equation (A.4) in GK Lemma 7(iii), the PDFs of the corresponding vectors $\boldsymbol{\Delta}^H \equiv (\Delta_1^H, \dots, \Delta_J^H)'$ and $\boldsymbol{\Delta}^L \equiv (\Delta_1^L, \dots, \Delta_J^L)'$ are asymptotically normal; specifically, uniformly over values $\mathbf{d}$ satisfying $|d_j| \le \log(n)$ for each $j = 1, \dots, J$,

$$f_{\boldsymbol{\Delta}^H}(\mathbf{d}) = \phi_{\underline{\mathcal{V}}^H}(\mathbf{d})\Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big],$$
$$f_{\boldsymbol{\Delta}^L}(\mathbf{d}) = \phi_{\underline{\mathcal{V}}^L}(\mathbf{d})\Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big],$$

where the row i, column k elements of covariance matrices $\underline{\mathcal{V}}^H$ and $\underline{\mathcal{V}}^L$ are, respectively,

$$\mathcal{V}_{i,k}^H = \min\big\{u_i^H(\tilde{\alpha}), u_k^H(\tilde{\alpha})\big\} - u_i^H(\tilde{\alpha})u_k^H(\tilde{\alpha}),$$
$$\mathcal{V}_{i,k}^L = \min\big\{u_i^L(\tilde{\alpha}), u_k^L(\tilde{\alpha})\big\} - u_i^L(\tilde{\alpha})u_k^L(\tilde{\alpha}).$$

Further, these PDFs are first-order equivalent. Let $\underline{\mathcal{V}}$ have row i, column k elements $\mathcal{V}_{i,k} = \min\{\tau_i, \tau_k\} - \tau_i\tau_k$. Then, $\mathcal{V}_{i,k}^H = \mathcal{V}_{i,k} + O(n^{-1/2})$ and $\mathcal{V}_{i,k}^L = \mathcal{V}_{i,k} + O(n^{-1/2})$ because both $u_j^H(\tilde{\alpha}) = \tau_j + O(n^{-1/2})$ and $u_j^L(\tilde{\alpha}) = \tau_j - O(n^{-1/2})$ by Lemma 6. Writing out the formula for a mean-zero multivariate normal PDF, with $|\cdot|$ denoting determinant, when $\mathbf{d} = O(\log(n))$,

$$
\begin{aligned}
f_{\boldsymbol{\Delta}^H}(\mathbf{d}) &= (2\pi)^{-J/2} \overbrace{\left|\underline{\mathcal{V}}^H\right|^{-1/2}}^{=\left|\underline{\mathcal{V}}+O(n^{-1/2})\right|^{-1/2}} \exp\bigl\{-(1/2)\mathbf{d}' \overbrace{(\underline{\mathcal{V}}^H)^{-1}}^{=\underline{\mathcal{V}}^{-1}+O(n^{-1/2})} \mathbf{d}\bigr\} \\
&= (2\pi)^{-J/2}\Bigl|\underline{\mathcal{V}}\Bigl[\underline{\mathbf{I}}_J + \underline{\mathcal{V}}^{-1}O(n^{-1/2})\Bigr]\Bigr|^{-1/2} \\
&\quad \times \exp\bigl\{-(1/2)\mathbf{d}'\underline{\mathcal{V}}^{-1}\mathbf{d} + O(\log(n))O(n^{-1/2})O(\log(n))\bigr\} \\
&= (2\pi)^{-J/2}|\underline{\mathcal{V}}|^{-1/2}\Bigl\{\Bigl[1+O(n^{-1/2})\Bigr]^J\Bigr\}^{-1/2} \\
&\quad \times \exp\bigl\{-(1/2)\mathbf{d}'\underline{\mathcal{V}}^{-1}\mathbf{d} + O(\log(n))O(n^{-1/2})O(\log(n))\bigr\} \\
&= (2\pi)^{-J/2}|\underline{\mathcal{V}}|^{-1/2}\Bigl[1+O(n^{-1/2})\Bigr]\exp\bigl\{-(1/2)\mathbf{d}'\underline{\mathcal{V}}^{-1}\mathbf{d}\bigr\}\overbrace{\exp\bigl\{O(n^{-1/2}[\log(n)]^2)\bigr\}}^{=1+O(n^{-1/2}[\log(n)]^2)} \\
&= (2\pi)^{-J/2}|\underline{\mathcal{V}}|^{-1/2}\exp\bigl\{-(1/2)\mathbf{d}'\underline{\mathcal{V}}^{-1}\mathbf{d}\bigr\}\Bigl[1+O\bigl(n^{-1/2}[\log(n)]^2\bigr)\Bigr] \\
&= \phi_{\underline{\mathcal{V}}}(\mathbf{d})\Bigl[1+O\bigl(n^{-1/2}[\log(n)]^2\bigr)\Bigr].
\end{aligned}
$$

The same applies to the PDF of $\boldsymbol{\Delta}^L$, so when each element of the argument is $O(\log(n))$,

$$
f_{\boldsymbol{\Delta}^H}(\mathbf{d}) = \phi_{\underline{\mathcal{V}}}(\mathbf{d})\Bigl[1+O\Bigl(n^{-1/2}[\log(n)]^2\Bigr)\Bigr], \quad f_{\boldsymbol{\Delta}^L}(\mathbf{d}) = \phi_{\underline{\mathcal{V}}}(\mathbf{d})\Bigl[1+O\Bigl(n^{-1/2}[\log(n)]^2\Bigr)\Bigr]. \tag{49}
$$

We also define

$$
D_j^H \equiv \sqrt{n}\bigl(u_j^H(\tilde{\alpha}) - \tau_j\bigr), \quad D_j^L \equiv \sqrt{n}\bigl(u_j^L(\tilde{\alpha}) - \tau_j\bigr), \tag{50}
$$

$$
D_j^L = \overbrace{-z_{1-\tilde{\alpha}}\sqrt{\tau_j(1-\tau_j)} - n^{-1/2}\frac{2\tau_j - 1}{6}(z_{1-\tilde{\alpha}}^2 + 2) + O(n^{-1})}^{\text{Lemma 6}} = -D_j^H + O(n^{-1/2}), \tag{51}
$$

$$
D_0^H \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_j^H, \quad D_0^L \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_j^L = -D_0^H + O(n^{-1/2}). \tag{52}
$$

Below, $\tilde{u}_j$ is between τ_j and $u_{0,j}^H$, so $\tilde{u}_j \to \tau_j$. Thus, for large enough n, all $\tilde{u}_j$ are in an arbitrarily small neighborhood of τ_j, and A2 implies $Q'''(\tilde{u}_j)$ is uniformly bounded. We have

$$
\sqrt{n}\Bigl(\tilde{Q}_U^I(u_{0,j}^H) - u_{0,j}^H\Bigr) = \overbrace{O\bigl([\log(n)]^{1/2}\bigr)}^{\text{from (47)}},
$$

$$
\begin{aligned}
\tilde{Q}_U^I(u_{0,j}^H) - \tau_j &= \Bigl[\sqrt{n}\Bigl(\tilde{Q}_U^I(u_{0,j}^H) - u_{0,j}^H\Bigr)\Bigr]n^{-1/2} + \overbrace{(u_{0,j}^H - \tau_j)}^{\text{Lemma 6}} \\
&= O\Bigl([\log(n)]^{1/2}n^{-1/2}\Bigr) + O(n^{-1/2}) = O\Bigl([\log(n)]^{1/2}n^{-1/2}\Bigr).
\end{aligned} \tag{53}
$$

For each j, the Taylor expansion is

$$Q\Big(\tilde{Q}^I_U(u^H_{0,j})\Big) - Q(\tau_j) = Q'(\tau_j)\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big] + \frac{1}{2}Q''(\tau_j)\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big]^2 \overbrace{+\frac{1}{6}\overbrace{Q'''(\tilde{u}_j)}^{=O(1)\text{ uniformly by A2}}\overbrace{\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big]^3}^{=O\left(n^{-3/2}[\log(n)]^{3/2}\right)\text{ by (53)}}}^{=O\left(n^{-3/2}[\log(n)]^{3/2}\right)} . \tag{54}$$

We continue from (44), making explicit the dependence on $\tilde{\alpha}$. Recall that for T_h, the true value of $\boldsymbol{\gamma}$ is used, so $\tilde{\alpha}$ is non-random, and $u^H_{0,j} = u^H_j(\tilde{\alpha})$. Plugging in (54), the third-order term can be pulled out using the (above) fact that $n^{1/2}\boldsymbol{\psi}'\big[Q\big(\tilde{Q}^I_U(\mathbf{u}^H_0)\big) - Q(\boldsymbol{\tau})\big]$ has an asymptotically normal PDF:

$$\begin{aligned}
T_h(\tilde{\alpha}) &= \mathrm{P}\left(\sum_{j=1}^J \psi_j \overbrace{\left\{Q'(\tau_j)\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big] + \frac{Q''(\tau_j)}{2}\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big]^2 + O\Big(n^{-3/2}[\log(n)]^{3/2}\Big)\right\}}^{\text{from (54)}}\right. \\
&\qquad \left. > 0\right) \\
&\quad - \mathrm{P}\left(\sum_{j=1}^J \psi_j Q'(\tau_j)\Big(\tilde{Q}^I_U\big(u^H_{0,j}\big) - \tau_j\Big) > 0\right) + O\Big(n^{-3/2}[\log(n)]^3\Big) \\
&= \mathrm{P}\left(\sum_{j=1}^J \psi_j Q'(\tau_j) \overbrace{\sqrt{n}\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big]}^{\text{non-degenerate normal PDF}}\right. \\
&\qquad \left. > -\sum_{j=1}^J \psi_j \frac{1}{2}Q''(\tau_j)\sqrt{n}\Big[\tilde{Q}^I_U(u^H_{0,j}) - \tau_j\Big]^2 - \overbrace{O\Big(n^{-1}[\log(n)]^{3/2}\Big)}^{\text{can pull out}}\right) \\
&\quad - \mathrm{P}\left(\sum_{j=1}^J \psi_j Q'(\tau_j)\Big(\tilde{Q}^I_U\big(u^H_{0,j}\big) - \tau_j\Big) > 0\right) \\
&\quad + O\Big(n^{-3/2}[\log(n)]^3\Big) \\
&= \overbrace{\mathrm{P}\left(\sum_{j=1}^J \psi_j Q'(\tau_j)\Big[\tilde{Q}^I_U\big(u^H_j(\tilde{\alpha})\big) - \tau_j\Big] > -\sum_{j=1}^J \psi_j \frac{1}{2}Q''(\tau_j)\Big[\tilde{Q}^I_U\big(u^H_j(\tilde{\alpha})\big) - \tau_j\Big]^2\right)}^{T_{H,1}} \\
&\quad \overbrace{- \mathrm{P}\left(\sum_{j=1}^J \psi_j Q'(\tau_j)\Big(\tilde{Q}^I_U\big(u^H_j(\tilde{\alpha})\big) - \tau_j\Big) > 0\right)}^{T_{H,2}} \\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big). \\
&= T_{H,1} - T_{H,2} + O\Big(n^{-1}[\log(n)]^{3/2}\Big),
\end{aligned}$$

$$T_{H,1} \equiv \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\big(\Delta_j^H + D_j^H\big) > -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j Q''(\tau_j)\big(\Delta_j^H + D_j^H\big)^2\right), \tag{55}$$

$$T_{H,2} \equiv \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\big(\Delta_j^H + D_j^H\big) > 0\right). \tag{56}$$

Following the same steps,

$$T_l(\tilde{\alpha}) = T_{L,1} - T_{L,2} + O(n^{-1}[\log(n)]^{3/2}), \tag{57}$$

$$T_{L,1} \equiv \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\big(\Delta_j^L + D_j^L\big) < -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j Q''(\tau_j)\big(\Delta_j^L + D_j^L\big)^2\right), \tag{58}$$

$$T_{L,2} \equiv \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\big(\Delta_j^L + D_j^L\big) < 0\right). \tag{59}$$

From (47), within $T_{H,1}$, $(\Delta_j^H + D_j^H)^2 = O\big(\log(n)\big)$, and $\psi_j Q''(\tau_j) = O(1)$. Thus, $T_{H,1} - T_{H,2}$ is the probability that the (non-degenerate) Gaussian random variable

$$\sum_{j=1}^{J} \psi_j Q'(\tau_j)\big(\Delta_j^H + D_j^H\big)$$

is between zero and

$$-\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j Q''(\tau_j)\big(\Delta_j^H + D_j^H\big)^2 = O\big(n^{-1/2}[\log(n)]\big).$$

The Gaussian PDF is $O(1)$, and it remains $O(1)$ after conditioning on the event that all $\Delta_j^2 \leq 2\log(n)$ (as in (47)) since the event's complement has smaller-order probability; heuristically,

$$f_{X|E}(x \mid E) = \mathbb{1}\{E\} f_X(x) / \mathrm{P}(E),$$

where $\mathrm{P}(E) = 1 - O\big(n^{-2}[\log(n)]^{-1/2}\big)$. Consequently, by the MVT, the probability of any interval of length $O\big(n^{-1/2}\log(n)\big)$ is $O\big(n^{-1/2}\log(n)\big)$:

$$\mathrm{P}(X \in [a,b] \mid E) = \int_a^b f_{X|E}(x \mid E)\,dx = \overbrace{[b-a]}^{=O\left(n^{-1/2}\log(n)\right)} \overbrace{f_{X|E}(\tilde{x} \mid E)}^{=O(1)}.$$

Altogether,

$$\begin{aligned} T_h &= T_{H,1} - T_{H,2} + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\ &= \overbrace{\big\{T_{H,1} - T_{H,2} \mid \text{any } (\Delta_j^H)^2 > 2\log(n)\big\}}^{\leq 1} \overbrace{\mathrm{P}\big(\text{any } (\Delta_j^H)^2 > 2\log(n)\big)}^{=O\left(n^{-2}[\log(n)]^{-1/2}\right) \text{ by } (48)} \end{aligned}$$

$$
\begin{aligned}
&\quad + \overbrace{\left\{T_{H,1} - T_{H,2} \mid \text{all } (\Delta_j^H)^2 \le 2\log(n)\right\}}^{=O\left(n^{-1/2}\log(n)\right)} \overbrace{\mathrm{P}\left(\text{all } (\Delta_j^H)^2 \le 2\log(n)\right)}^{\le 1} \\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= O\Big(n^{-2}[\log(n)]^{-1/2} + n^{-1/2}\log(n) + n^{-1}[\log(n)]^{3/2}\Big) \\
&= O\Big(n^{-1/2}\log(n)\Big).
\end{aligned}
$$

The orders of the terms for T_l are identical, so the same result follows.

For the two-sided result, we must be more precise. The strategy is to consider the inside of $T_{H,1}$ as a quadratic in Δ_1^H conditional on $\Delta_2^H, \ldots, \Delta_J^H$. The "negative" root is proportional to $n^{1/2}$, so (still conditional on the other Δ_j^H) $T_{H,1}$ can be approximated as the probability that Δ_1^H is above the other root. These conditional probabilities are then integrated over the distribution of the other Δ_j^H, which are asymptotically jointly normal. Again, we restrict attention to when the $|\Delta_j^H| \le \sqrt{2\log(n)}$, which is violated with asymptotically negligible probability. Using a similar argument for $T_{L,1}$ and using the symmetry from, e.g., Lemma 6 will then complete the proof.

We first rewrite $T_{H,2}$. Without loss of generality, let $\psi_1 = 1$. (If $\psi_1 \neq 1$, then letting $\tilde{\psi}_j = \psi_j/\psi_1$ achieves $\tilde{\psi}_1 = 1$, and the final confidence interval can be multiplied by ψ_1 to reverse the transformation.) Also, let $\gamma_j = Q'(\tau_j)/Q'(\tau_1)$, so $\gamma_1 = 1$. Let

$$
\boldsymbol{\Delta}_{-1}^H \equiv \left(\Delta_2^H, \ldots, \Delta_J^H\right)', \quad \boldsymbol{\Delta}_{-1}^L \equiv \left(\Delta_2^L, \ldots, \Delta_J^L\right)'. \tag{60}
$$

Define the function $\pi^{H,2}(\cdot) : \mathbb{R}^{J-1} \mapsto \mathbb{R}$ as

$$
\pi^{H,2}(\mathbf{v}_{-1}) = -D_0^H - \sum_{j=2}^{J} \psi_j \gamma_j v_j \tag{61}
$$

for any argument $\mathbf{v}_{-1} = (v_2, \ldots, v_J)' \in \mathbb{R}^{J-1}$, to help simplify $T_{H,2}$ as follows. Starting from (56), now we can rewrite $T_{H,2}$:

$$
\begin{aligned}
T_{H,2} &= \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\left(\Delta_j^H + D_j^H\right) > 0\right) \\
&= \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \gamma_j \left(\Delta_j^H + D_j^H\right) > 0\right) \\
&= \mathrm{P}\left(\Delta_1^H + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^H > -D_0^H\right) \\
&= \mathrm{P}\left(\Delta_1^H > \pi^{H,2}(\boldsymbol{\Delta}_{-1}^H)\right).
\end{aligned} \tag{62}
$$

Starting from (55),

$$
T_{H,1} = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\left(\Delta_j^H + D_j^H\right) > -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j Q''(\tau_j)\left(\Delta_j^H + D_j^H\right)^2\right)
$$

$$
\begin{aligned}
&= \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \gamma_j \big(\Delta_j^H + D_j^H\big) > -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \big(\Delta_j^H + D_j^H\big)^2 \right) \\
&= \mathrm{P}\Bigg(D_0^H + \Delta_1^H + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^H > -\frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} \big[(\Delta_1^H)^2 + 2\Delta_1^H D_1^H + (D_1^H)^2\big] \\
&\qquad\qquad - \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \big(\Delta_j^H + D_j^H\big)^2 \Bigg) \\
&= \mathrm{P}\Bigg((\Delta_1^H)^2 \left[\frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} \right] + \Delta_1^H \left[1 + n^{-1/2} D_1^H \frac{Q''(\tau_1)}{Q'(\tau_1)} \right] \\
&\qquad + \left[D_0^H + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^H + \frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} (D_1^H)^2 + \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \big(\Delta_j^H + D_j^H\big)^2 \right] \\
&\qquad > 0 \Bigg) \\
&= \mathrm{P}\big(a(\Delta_1^H)^2 + b\Delta_1^H + c > 0 \big), \\
&a \equiv n^{-1/2} a_0, \quad a_0 \equiv \frac{Q''(\tau_1)}{2Q'(\tau_1)}, \\
&b \equiv 1 + n^{-1/2} b_0, \quad b_0 \equiv D_1^H \frac{Q''(\tau_1)}{Q'(\tau_1)}, \quad b^{-1} = 1 - n^{-1/2} b_0 + O(n^{-1}), \\
&c \equiv -\pi^{H,2}(\boldsymbol{\Delta}_{-1}^H) + \frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} (D_1^H)^2 + \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \big(\Delta_j^H + D_j^H\big)^2 \\
&\quad = O\Big([\log(n)]^{1/2} + n^{-1/2} + n^{-1/2} \log(n) \Big) = O\Big([\log(n)]^{1/2} \Big).
\end{aligned}
$$

It's possible that $a = 0$, if (and only if) $a_0 = 0$, which occurs iff $Q''(\tau_1) = 0$ (e.g., if $\tau_1 = 0.5$ and the population is normal). In that case, the probability simplifies to

$$
\mathrm{P}\big(\Delta_1^H > -c/b\big). \tag{63}
$$

As noted below, this results in an equivalent expression (up to smaller-order terms) to the general case below.

The roots of $ax^2 + bx + c = 0$ are

$$
r_- \equiv \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad r_+ \equiv \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{64}
$$

The subscript refers only to the $\pm$ in the quadratic formula; we may have either $r_- < r_+$ or $r_- > r_+$ depending on the sign of a. Both roots are real (for large enough n) since $b^2 \approx 1$ while $4ac = O\big(n^{-1/2}[\log(n)]^{1/2}\big) \to 0$. Since $a = O(n^{-1/2})$, we approximate $g(a) = \sqrt{b^2 - 4ac}$ around $a = 0$:

$$
\begin{aligned}
g'(a) &= (1/2)(b^2 - 4ac)^{-1/2}(-4c), \quad g''(a) = (-1/4)(b^2 - 4ac)^{-3/2}(-4c)^2, \\
g'''(a) &= (3/8)(b^2 - 4ac)^{-5/2}(-4c)^3, \\
g(a) &= g(0) + ag'(0) + (1/2)a^2 g''(0) + (1/6)a^3 g'''(\tilde{a}),
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{b^2-4ac} &= b + a(1/2)b^{-1}(-4c) + (1/2)a^2(-1/4)b^{-3}(-4c)^2 + O\big(n^{-3/2}[\log(n)]^{3/2}\big),\\
r_+ &= \frac{-b+b-2ac/b-2a^2c^2/b^3+O\big(n^{-3/2}[\log(n)]^{3/2}\big)}{2a} = -\frac{c}{b} - \frac{ac^2}{b^3} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= -c(1-n^{-1/2}b_0) - n^{-1/2}a_0[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^H)]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big).
\end{aligned}
$$

Similar to (61), define the function $\pi^{H,1}(\cdot) : \mathbb{R}^{J-1} \mapsto \mathbb{R}$ so that

$$
r_+ = \pi^{H,1}(\boldsymbol{\Delta}_{-1}^H) = -c(1-n^{-1/2}b_0) - n^{-1/2}a_0[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^H)]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big), \tag{65}
$$

where c also depends implicitly on the argument.

For r_-,

$$
\begin{aligned}
r_- &= \frac{-b-b+2ac/b+2a^2c^2/b^3+O\big(n^{-3/2}[\log(n)]^{3/2}\big)}{2a}\\
&= -b/a + c/b + ac^2/b^3 + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= -n^{1/2}(1+n^{-1/2}b_0)/a_0 + O([\log(n)]^{1/2}) + O\big(n^{-1/2}\log(n)\big) + O\Big(n^{-3/2}[\log(n)]^{3/2}\Big)\\
&= -n^{1/2}/a_0 + O([\log(n)]^{1/2}).
\end{aligned}
$$

If $a_0 = 0$, then there is essentially only r_+, as seen in (63). If $a_0 \neq 0$, then $r_- \asymp n^{1/2}$ since a_0 is a finite, fixed constant by Assumption A2.

If $a > 0$, then $r_- < r_+$ and the function $ax^2 + bx + c$ is positive when either $x < r_-$ or $x > r_+$:

$$
\begin{aligned}
T_{H,1} &= \mathrm{P}\big(a(\Delta_1^H)^2 + b\Delta_1^H + c > 0\big)\\
&= \mathrm{P}\big(\Delta_1^H > r_+\big) + \mathrm{P}\big(\Delta_1^H < r_-\big).
\end{aligned}
$$

Conditional on the other Δ_j^H, even if they are all as big as allowed by (47), they are only of order $\sqrt{\log(n)}$ while r_- is of polynomial $n^{1/2}$ order (and the variance of Δ_1^H is still $O(1)$), so we always have an exponentially small $\mathrm{P}\big(\Delta_1^H < r_-\big) = O\big(e^{-0.99n}\big)$, to give a loose but sufficiently small bound.

If $a < 0$, then $r_+ < r_-$ and the function $ax^2 + bx + c$ is positive iff $r_+ < x < r_-$, so

$$
\begin{aligned}
T_{H,1} &= \mathrm{P}\big(a(\Delta_1^H)^2 + b\Delta_1^H + c > 0\big)\\
&= \mathrm{P}\big(\Delta_1^H > r_+\big) - \mathrm{P}\big(\Delta_1^H > r_-\big).
\end{aligned}
$$

Again, given any $\boldsymbol{\Delta}_{-1}^H$ satisfying the bound in (47), the $\mathrm{P}\big(\Delta_1^H > r_-\big)$ term is exponentially small. Thus, regardless of $a > 0$ or $a < 0$,

$$
T_{H,1} = \mathrm{P}\big(\Delta_1^H > r_+\big) + O\big(e^{-0.99n}\big). \tag{66}
$$

When $a = 0$, (63) had $\mathrm{P}\big(\Delta_1^H > -c/b\big)$. This is the same as above but without the exponentially small error and without the $-ac^2/b^3$ or remainder term inside r_+.

To fully treat $T_{H,1}$, we must account for the fact that r_+ contains random variables, by integrating a conditional probability over the distribution of the conditioning variable, $\boldsymbol{\Delta}_{-1}^{H}$:

$$T_{H,1} = \mathbb{E}_{\boldsymbol{\Delta}_{-1}^{H}}\left[\mathrm{P}\left(\Delta_1^H > r_+ \mid \boldsymbol{\Delta}_{-1}^{H}\right)\right] = \underset{v_j^2 \leq 2\log(n),\, j\geq 2}{\int \cdots \int} \int_{r_+}^{\infty} f_{\Delta_1^H \mid \boldsymbol{\Delta}_{-1}^{H}}(v_1 \mid \mathbf{v}_{-1}) f_{\boldsymbol{\Delta}_{-1}^{H}}(\mathbf{v}_{-1})\, dv_1\, dv_2 \ldots dv_J. \tag{67}$$

Overall,

$$\begin{aligned} T_h &= T_{H,1} - T_{H,2} + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\ &= \overbrace{\mathrm{P}\big(\Delta_1^H > \pi^{H,1}(\boldsymbol{\Delta}_{-1}^{H})\big) + O\big(e^{-0.99n}\big)}^{\text{from (65) and (66)}} - \overbrace{\mathrm{P}\big(\Delta_1^H > \pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})\big)}^{\text{from (62)}} + O\Big(n^{-1}[\log(n)]^{3/2}\Big), \end{aligned} \tag{68}$$

so we need the probability that Δ_1^H is between $\pi^{H,1}(\boldsymbol{\Delta}_{-1}^{H})$ and $\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})$. Toward that end,

$$\begin{aligned} &\overbrace{\pi^{H,1}(\boldsymbol{\Delta}_{-1}^{H})}^{\text{from (65)}} - \pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H}) \\ &= -c(1 - n^{-1/2}b_0) - n^{-1/2}a_0[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big) - \pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H}) \\ &= \overbrace{\left[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H}) - \frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}(D_1^H)^2 - \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)}(\Delta_j^H + D_j^H)^2\right]}^{-c} \\ &\quad + n^{-1/2}\overbrace{\Big[-\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H}) + O\Big(n^{-1/2}\log(n)\Big)\Big]}^{c} b_0 - n^{-1/2}a_0[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})]^2 \\ &\quad - \pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H}) + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\ &= -\frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}(D_1^H)^2 - \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)}(\Delta_j^H + D_j^H)^2 \\ &\quad - n^{-1/2}\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})b_0 - n^{-1/2}a_0\big[\pi^{H,2}(\boldsymbol{\Delta}_{-1}^{H})\big]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big). \end{aligned}$$

Plugging in for a_0 and b_0, and emphasizing that $\pi^{H,1}(\cdot)$ and $\pi^{H,2}(\cdot)$ are functions taking any argument $\mathbf{v}_{-1}$,

$$\begin{aligned} &\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1}) \\ &= -\big[\pi^{H,1}(\mathbf{v}_{-1}) - \pi^{H,2}(\mathbf{v}_{-1})\big] \\ &= \frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}(D_1^H)^2 + \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)}(v_j + D_j^H)^2 \\ &\quad + n^{-1/2}\pi^{H,2}(\mathbf{v}_{-1})D_1^H \frac{Q''(\tau_1)}{Q'(\tau_1)} + n^{-1/2}\frac{Q''(\tau_1)}{Q'(\tau_1)}\big[\pi^{H,2}(\mathbf{v}_{-1})\big]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\ &= O\Big(n^{-1/2}\log(n)\Big). \end{aligned}$$

Then, using (67) and (68), letting $\underline{\mathcal{V}}_{-1}$ be the $(J-1)\times(J-1)$ lower-right submatrix of $\underline{\mathcal{V}}$, and letting $\phi_{V_1\mid\mathbf{V}_{-1}}(v_1 \mid \mathbf{v}_{-1})$ denote the conditional PDF of V_1 given $\mathbf{V}_{-1}$ for random vector $\mathbf{V}$ with

PDF $\phi_{\underline{\mathcal{V}}}(\cdot)$,

$$\begin{aligned}
T_h &= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \int_{\pi^{H,1}(\mathbf{v}_{-1})}^{\pi^{H,2}(\mathbf{v}_{-1})} f_{\Delta_1^H \mid \boldsymbol{\Delta}_{-1}^H}(v_1 \mid \mathbf{v}_{-1}) f_{\boldsymbol{\Delta}_{-1}^H}(\mathbf{v}_{-1})\, dv_1\, dv_2 \ldots dv_J + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \int_{\pi^{H,1}(\mathbf{v}_{-1})}^{\pi^{H,2}(\mathbf{v}_{-1})} f_{\boldsymbol{\Delta}^H}(\mathbf{v})\, dv_1\, dv_2 \ldots dv_J + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \int_{\pi^{H,1}(\mathbf{v}_{-1})}^{\pi^{H,2}(\mathbf{v}_{-1})} \overbrace{\phi_{\underline{\mathcal{V}}}(\mathbf{v})\Big[1 + O\Big(n^{-1/2}\log(n)\Big)\Big]}^{\text{by (49) but with } x = O\left([\log(n)]^{1/2}\right)}\, dv_1\, dv_2 \ldots dv_J + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \int_{\pi^{H,1}(\mathbf{v}_{-1})}^{\pi^{H,2}(\mathbf{v}_{-1})} \phi_{V_1 \mid \mathbf{V}_{-1}}(v_1 \mid \mathbf{v}_{-1}) \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_1\, dv_2 \ldots dv_J \Big[1 + O\Big(n^{-1/2}\log(n)\Big)\Big] \\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \overbrace{\big[\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1})\big] \phi_{V_1 \mid \mathbf{V}_{-1}}(\tilde{v}_1 \mid \mathbf{v}_{-1})}^{\text{mean value theorem}} \\
&\qquad\qquad \times \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_2 \ldots dv_J \Big[1 + O\Big(n^{-1/2}\log(n)\Big)\Big] \\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \overbrace{\big[\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1})\big]}^{=O\left(n^{-1/2}\log(n)\right)} \Big[\phi_{V_1 \mid \mathbf{V}_{-1}}\big(\pi^{H,2}(\mathbf{v}_{-1}) \mid \mathbf{v}_{-1}\big) + O\big(n^{-1/2}\log(n)\big)\Big] \\
&\qquad\qquad \times \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_2 \ldots dv_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big) \\
&= \underset{v_j^2 \le 2\log(n),\, j\ge 2}{\int\cdots\int} \big[\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1})\big] \phi_{V_1 \mid \mathbf{V}_{-1}}\big(\pi^{H,2}(\mathbf{v}_{-1}) \mid \mathbf{v}_{-1}\big) \phi_{\underline{\mathcal{V}}}(\mathbf{v}_{-1})\, dv_2 \ldots dv_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big).
\end{aligned} \tag{69}$$

For T_l, the results are parallel, eventually leading to $T_l = -T_h + O\big(n^{-1}[\log(n)]^2\big)$ in (73). Returning to (57),

$$\begin{aligned}
T_l &= T_{L,1} - T_{L,2} + O\Big(n^{-1}[\log(n)]^{3/2}\Big), \\
T_{L,2} &= \mathrm{P}\left(\Delta_1^L + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^L < -D_0^L\right) \\
&= \mathrm{P}\big(\Delta_1^L < \pi^{L,2}(\boldsymbol{\Delta}_{-1}^L)\big),
\end{aligned}$$

$$\pi^{L,2}(\mathbf{v}_{-1}) \equiv -\sum_{j=2}^{J} \psi_j \gamma_j v_j - D_0^L,$$

$$\begin{aligned}
T_{L,1} &= \mathrm{P}\left(\sum_{j=1}^{J} \psi_j Q'(\tau_j)\bigl(\Delta_j^L + D_j^L\bigr) < -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j Q''(\tau_j)\bigl(\Delta_j^L + D_j^L\bigr)^2 \right) \\
&= \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \gamma_j \bigl(\Delta_j^L + D_j^L\bigr) < -\frac{n^{-1/2}}{2} \sum_{j=1}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \bigl(\Delta_j^L + D_j^L\bigr)^2 \right) \\
&= \mathrm{P}\Biggl(D_0^L + \Delta_1^L + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^L < -\frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} \bigl[(\Delta_1^L)^2 + 2\Delta_1^L D_1^L + (D_1^L)^2\bigr] \\
&\qquad - \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \bigl(\Delta_j^L + D_j^L\bigr)^2 \Biggr) \\
&= \mathrm{P}\Biggl((\Delta_1^L)^2 \left[\frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} \right] + \Delta_1^L \left[1 + n^{-1/2} D_1^L \frac{Q''(\tau_1)}{Q'(\tau_1)} \right] \\
&\qquad + \Biggl[D_0^L + \sum_{j=2}^{J} \psi_j \gamma_j \Delta_j^L + \frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} (D_1^L)^2 \\
&\qquad + \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \bigl(\Delta_j^L + D_j^L\bigr)^2 \Biggr] \\
&\qquad < 0 \Biggr) \\
&= \mathrm{P}\bigl(a(\Delta_1^L)^2 + b\Delta_1^L + c < 0\bigr), \\
a &\equiv n^{-1/2} a_0, \quad a_0 \equiv \frac{Q''(\tau_1)}{2Q'(\tau_1)}, \\
b &\equiv 1 + n^{-1/2} b_0, \quad b_0 \equiv D_1^L \frac{Q''(\tau_1)}{Q'(\tau_1)}, \\
c &\equiv -\pi^{L,2}(\boldsymbol{\Delta}_{-1}^L) + \frac{n^{-1/2}}{2} \frac{Q''(\tau_1)}{Q'(\tau_1)} (D_1^L)^2 + \frac{n^{-1/2}}{2} \sum_{j=2}^{J} \psi_j \frac{Q''(\tau_j)}{Q'(\tau_1)} \bigl(\Delta_j^L + D_j^L\bigr)^2,
\end{aligned}$$

with rates identical to those for $T_{H,1}$. Thus, the arguments concerning the roots of the quadratic are identical, only changing $>$ to $<$:

$$\begin{aligned}
T_{L,1} &= \mathrm{P}\bigl(\Delta_1^L < \pi^{L,1}(\boldsymbol{\Delta}_{-1}^L)\bigr) + O\bigl(e^{-0.99n}\bigr), \\
\pi^{L,1}(\boldsymbol{\Delta}_{-1}^L) &\equiv -c\Bigl(1 - n^{-1/2} b_0\Bigr) - n^{-1/2} a_0 \bigl[\pi^{L,2}(\boldsymbol{\Delta}_{-1}^L)\bigr]^2 + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr).
\end{aligned}$$

Since the inequality is the opposite direction from T_h, instead of $\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1})$ we compute

$$\begin{aligned}
&\pi^{L,1}(\mathbf{v}_{-1}) - \pi^{L,2}(\mathbf{v}_{-1}) \\
&= -c\Bigl(1 - n^{-1/2} b_0\Bigr) - n^{-1/2} a_0 \bigl[\pi^{L,2}(\mathbf{v}_{-1})\bigr]^2 + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr) - \pi^{L,2}(\mathbf{v}_{-1})
\end{aligned}$$

$$= -\frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}(D_1^L)^2 - \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\frac{Q''(\tau_j)}{Q'(\tau_1)}(v_j + D_j^L)^2$$
$$- n^{-1/2}\pi^{L,2}(\mathbf{v}_{-1})D_1^L\frac{Q''(\tau_1)}{Q'(\tau_1)} - n^{-1/2}\big[\pi^{L,2}(\mathbf{v}_{-1})\big]^2\frac{Q''(\tau_1)}{2Q'(\tau_1)} + O\Big(n^{-1}[\log(n)]^{3/2}\Big).$$

Repeating the steps in the derivation of (69) yields

$$T_l = \int\cdots\int_{v_j^2\le 2\log(n),\, j\ge 2}\big[\pi^{L,1}(\mathbf{v}_{-1}) - \pi^{L,2}(\mathbf{v}_{-1})\big]\phi_{V_1|\mathbf{V}_{-1}}\big(\pi^{L,2}(\mathbf{v}_{-1}) \mid \mathbf{v}_{-1}\big)\phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\,dv_2\ldots dv_J$$
$$+ O\big(n^{-1}[\log(n)]^2\big). \tag{70}$$

Since $D_0^L = -D_0^H + O(n^{-1/2})$,

$$\pi^{L,2}(-\mathbf{v}_{-1}) = -\pi^{H,2}(\mathbf{v}_{-1}) + O(n^{-1/2}). \tag{71}$$

Also,

$$\pi^{L,1}(-\mathbf{v}_{-1}) - \pi^{L,2}(-\mathbf{v}_{-1})$$
$$= -\frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}(D_1^L)^2 - \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\frac{Q''(\tau_j)}{Q'(\tau_1)}\big[v_j^2 - 2v_jD_j^L + (D_j^L)^2\big]$$
$$- n^{-1/2}\pi^{L,2}(-\mathbf{v}_{-1})D_1^L\frac{Q''(\tau_1)}{Q'(\tau_1)} - n^{-1/2}\big[\pi^{L,2}(-\mathbf{v}_{-1})\big]^2\frac{Q''(\tau_1)}{2Q'(\tau_1)} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)$$
$$= -\frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}\Big[-D_1^H + O(n^{-1/2})\Big]^2$$
$$- \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\frac{Q''(\tau_j)}{Q'(\tau_1)}\left[v_j^2 - 2v_j\Big(-D_j^H + O(n^{-1/2})\Big) + \Big(-D_j^H + O(n^{-1/2})\Big)^2\right]$$
$$- n^{-1/2}\Big[-\pi^{H,2}(\mathbf{v}_{-1}) + O(n^{-1/2})\Big]\Big[-D_1^H + O(n^{-1/2})\Big]\frac{Q''(\tau_1)}{Q'(\tau_1)}$$
$$- n^{-1/2}\Big[-\pi^{H,2}(\mathbf{v}_{-1}) + O(n^{-1/2})\Big]^2\frac{Q''(\tau_1)}{2Q'(\tau_1)} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)$$
$$= -\frac{n^{-1/2}Q''(\tau_1)}{2Q'(\tau_1)}\big(D_1^H\big)^2 - \frac{n^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\frac{Q''(\tau_j)}{Q'(\tau_1)}\Big[v_j^2 + 2v_jD_j^H + \big(D_j^H\big)^2\Big]$$
$$- n^{-1/2}\pi^{H,2}(\mathbf{v}_{-1})D_1^H\frac{Q''(\tau_1)}{Q'(\tau_1)} - n^{-1/2}\big[\pi^{H,2}(\mathbf{v}_{-1})\big]^2\frac{Q''(\tau_1)}{2Q'(\tau_1)} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)$$
$$= \pi^{H,1}(\mathbf{v}_{-1}) - \pi^{H,2}(\mathbf{v}_{-1}) + O\Big(n^{-1}[\log(n)]^{3/2}\Big). \tag{72}$$

Because a mean-zero normal distribution is symmetric about zero, and the area of integration is also symmetric about zero, we can replace $\mathbf{v}_{-1}$ with $-\mathbf{v}_{-1}$. Heuristically, in one dimension, with the change of variables $y = -x$,

$$\int_a^b g(x)\phi(x)\,dx = \int_{-a}^{-b} g(-y)\phi(-y)\,[-dy] = \int_{-b}^{-a} g(-y)\phi(y)\,dy,$$

and if $b = -a$ then $[-b, -a] = [a, b]$. Thus, continuing,

$$
\begin{aligned}
T_l &= \int \cdots \int_{v_j^2 \le 2\log(n),\, j \ge 2} \overbrace{\big[\pi^{L,1}(-\mathbf{v}_{-1}) - \pi^{L,2}(-\mathbf{v}_{-1})\big]}^{\text{apply (72)}} \overbrace{\phi_{V_1 \mid \mathbf{V}_{-1}}\big(\pi^{L,2}(-\mathbf{v}_{-1}) \mid -\mathbf{v}_{-1}\big)}^{\text{apply (71)}} \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_2 \dots dv_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big) \\
&= \int \cdots \int_{v_j^2 \le 2\log(n),\, j \ge 2} \Big[\pi^{H,1}(\mathbf{v}_{-1}) - \pi^{H,2}(\mathbf{v}_{-1}) + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\Big] \\
&\qquad\qquad \times \phi_{V_1 \mid \mathbf{V}_{-1}}\Big(-\pi^{H,2}(\mathbf{v}_{-1}) + O(n^{-1/2}) \mid -\mathbf{v}_{-1}\Big) \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_2 \dots dv_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big) \\
&= -\int \cdots \int_{v_j^2 \le 2\log(n),\, j \ge 2} \big[\pi^{H,2}(\mathbf{v}_{-1}) - \pi^{H,1}(\mathbf{v}_{-1})\big] \phi_{V_1 \mid \mathbf{V}_{-1}}\big(\pi^{H,2}(\mathbf{v}_{-1}) \mid \mathbf{v}_{-1}\big) \phi_{\underline{\mathcal{V}}_{-1}}(\mathbf{v}_{-1})\, dv_2 \dots dv_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big) \\
&= -T_h + O\big(n^{-1}[\log(n)]^2\big)
\end{aligned} \tag{73}
$$

since $\phi_{V_1 \mid \mathbf{V}_{-1}}(v_1 \mid \mathbf{v}_{-1}) = \phi_{V_1 \mid \mathbf{V}_{-1}}(-v_1 \mid -\mathbf{v}_{-1})$ by symmetry, using the formula for the conditional distribution from a mean-zero multivariate normal distribution. Thus,

$$
T_h(\tilde{\alpha}) + T_l(\tilde{\alpha}) = T_h(\tilde{\alpha}) + \big[-T_h(\tilde{\alpha}) + O\big(n^{-1}[\log(n)]^2\big)\big] = O\big(n^{-1}[\log(n)]^2\big)
$$

for any $\tilde{\alpha}$ in the range of possible values (which is bounded away from zero and fixed as $n \to \infty$). □

A.3.2 CPE from nuisance parameter estimation error: E_h, E_l

Lemma 9. *Under the assumptions of Theorem 7,* $E_h = O\big(m^{-1}\log(n) + (m/n)^2\big)$ *in (43), and similarly for the corresponding upper one-sided term,* $E_l = O\big(m^{-1}\log(n) + (m/n)^2\big)$*, where* m *is the common rate of smoothing parameters,* $m_j \asymp m$ *for all* j*.*

Proof. We continue to use notation from (28). As in (28), let $\mathbf{Y}^{\boldsymbol{\tau}} \equiv \Big(U_{n:\lfloor (n+1)\tau_j \rfloor}\Big)_{j=1}^{J}$, with Ω_j^+ and Ω_j^- defined with respect to this $\mathbf{Y}^{\boldsymbol{\tau}}$ as in (29). For simplicity, we write m instead of m_j since all m_j have the same rate. We return to using $\boldsymbol{\gamma} \equiv Q'(\boldsymbol{\tau})$ instead of the normalized version.

We continue to assume Condition $\star$ holds for realizations of all random variables, so

$$
\mathbf{Y}^{\boldsymbol{\tau}} - \boldsymbol{\tau} = O\Big(n^{-1/2}\log(n)\Big), \quad \boldsymbol{\Omega} - 2m/(n+1) = O\Big(n^{-1}m^{1/2}\log(n)\Big) \implies \boldsymbol{\Omega} = O(m/n)
$$

since $\log(n) \lesssim m^{1/2}$ as assumed in Theorem 7.

We start from the sparsity estimator in (12), using a mean value expansion where $\tilde{Y}_j$ and $\tilde{\tilde{Y}}_j$ are between $Y_j - \Omega_j^-$ and $Y_j + \Omega_j^+$. Under Condition $\star$, $\tilde{Y}_j \to \tau_j$, so Assumption A2 implies $Q'(\tilde{Y}_j)$ and

$Q''(\tilde{Y}_j)$ exist and are uniformly bounded for large enough n, and similarly for $\tilde{\tilde{Y}}_j$. So,[24]

$$
\begin{aligned}
\widehat{Q'}(\tau_j) &= \frac{n}{2m}\left[X_{n:\lfloor(n+1)\tau_j\rfloor+m_j} - X_{n:\lfloor(n+1)\tau_j\rfloor-m_j}\right] \\
&= \frac{n}{2m}\left[Q\left(Y_j+\Omega_j^+\right) - Q\left(Y_j-\Omega_j^-\right)\right] \\
&= \frac{n}{2m}\left[Q'(Y_j)\Omega_j + (1/2)Q''(\tilde{Y}_j)\Omega_j^2\right] \\
&= \frac{n}{2m}\Omega_j\left[Q'(\tau_j) + Q''(\tilde{\tilde{Y}}_j)(Y_j-\tau_j)\right] + O\big((n/m)\Omega_j^2\big) \\
&= \frac{n}{2m}\left[2m/(n+1) + O\left(n^{-1}m^{1/2}\log(n)\right)\right]\left[Q'(\tau_j) + Q''(\tilde{\tilde{Y}}_j)(Y_j-\tau_j)\right] \\
&\quad + O\big((n/m)(m/n)^2\big) \\
&= \left[1 + O\left(m^{-1/2}\log(n)\right)\right]\left[Q'(\tau_j) + O(1)O\left(n^{-1/2}\log(n)\right)\right] + O(m/n) \\
&= Q'(\tau_j) + O\left(m^{-1/2}\log(n)\right) + O\left(n^{-1/2}\log(n)\right) + O(m/n) \\
&= Q'(\tau_j) + O\big(m^{-1/2}\log(n) + m/n\big).
\end{aligned} \tag{74}
$$

From Lemma 6, where the $O(n^{-1})$ remainder is uniform since $\tilde{\alpha}$ is bounded away from zero and thus $z_{1-\tilde{\alpha}}$ is bounded,

$$
\begin{aligned}
\frac{d}{d\tilde{\alpha}}u_j^h(\tilde{\alpha}) &= \frac{d}{d\tilde{\alpha}}\left[\tau_j + n^{-1/2}\Phi^{-1}(1-\tilde{\alpha})\sqrt{\tau_j(1-\tau_j)} - \frac{2\tau_j-1}{6n}(z_{1-\tilde{\alpha}}^2+2) + O(n^{-3/2})\right] \\
&= 0 - n^{-1/2}\frac{\sqrt{\tau_j(1-\tau_j)}}{\phi(\Phi^{-1}(1-\tilde{\alpha}))} + O(n^{-1}) = O(n^{-1/2}),
\end{aligned} \tag{75}
$$

and the order is the same for the derivative of $u_j^l(\tilde{\alpha})$.

We also will need the derivative of the function $\tilde{\alpha}(\cdot)$ to be $O(1)$, with generic argument $\mathbf{g}$. In the subsequent application, $\tilde{g}_j$ is a value $\tilde{Q}'(\tau_j)$ between the true $Q'(\tau_j)$ and estimated $\widehat{Q'}(\tau_j)$ that arises from the MVT. Consequently, restrictions on the range of possible values $\widehat{Q'}(\tau_j)$ under Condition $\star$ apply to $\tilde{g}_j$, too. Now, the result that $\frac{\partial\tilde{\alpha}}{\partial g_j}$ evaluated at $\tilde{g}_j$ is uniformly bounded for all j can be shown with the implicit function theorem. Let P denote (temporarily) the RHS of (36) or (37), so

$$\frac{\partial\tilde{\alpha}}{\partial g_j} = -\frac{\partial P/\partial g_j}{\partial P/\partial\tilde{\alpha}}.$$

Thus, it suffices to show that $\frac{\partial P}{\partial g_j} = O(1)$ uniformly over possible $\tilde{g}_j$ values and that $\frac{\partial P}{\partial\tilde{\alpha}}$ is (uniformly) bounded away from zero. Heuristically, note that $g_j \in (0,\infty)$ while $\tilde{\alpha} \in [\alpha, 1]$, and asymptotically, (near) singularity points occur with smaller-order probability, so it is intuitive that $\frac{\partial\tilde{\alpha}}{\partial g_j}$ is bounded.

To the first order, $P = \mathrm{P}(n^{-1/2}Z < 0) = \mathrm{P}(Z < 0)$ (or > 0) where $Z \overset{a}{\sim} \mathrm{N}\big(\mu(\mathbf{g},\tilde{\alpha}), \sigma^2(\mathbf{g},\tilde{\alpha})\big)$, since Z is a linear combination of (asymptotically) jointly normal random variables with means and

[24] For comparison, from equations (2.5) and (2.6) in Bloch and Gastwirth (1968), the bias and standard deviation of $\widehat{Q'}(\tau_j)$ are respectively $O(m^2/n^2)$ and $O(m^{-1/2})$, so $\widehat{Q'}(\tau_j) = O_p\left(m^{-1/2} + (m/n)^2\right)$. The m^2/n^2 comes from the next term in the Taylor approximation because the second-order term above zeroes out when taking an expectation.

variances depending on $\mathbf{g}$ and $\tilde{\alpha}$. More explicitly, (37) is

$$
\begin{aligned}
0 &= \mathrm{P}\Bigg(\sum_{j=1}^{J}\psi_j g_j \overbrace{\sqrt{n}\Big[\tilde{Q}_U^I\big(u_j^L(\tilde{\alpha})\big)-\tau_j\Big]}^{\equiv Z_j \overset{a}{\sim}\text{Normal}} < 0\Bigg) - (1-\alpha) \\
&= \mathrm{P}\big(\qquad \overbrace{\mathbf{c}'\mathbf{Z}}^{=Z\overset{a}{\sim}\mathrm{N}(\mu(\mathbf{g},\tilde{\alpha}),\sigma^2(\mathbf{g},\tilde{\alpha}))} \qquad < 0\big) - (1-\alpha) \\
&\doteq \Phi\left(-\frac{\mu(\mathbf{g},\tilde{\alpha})}{\sigma(\mathbf{g},\tilde{\alpha})}\right) - (1-\alpha),
\end{aligned}
$$

where the linear combination weights are $c_j = \psi_j g_j$. Using (46) and (50), the normality is from

$$
\begin{aligned}
\sqrt{n}\Big[\tilde{Q}_U^I\big(\mathbf{u}^L(\tilde{\alpha})\big)-\boldsymbol{\tau}\Big] &= \boldsymbol{\Delta}^L + \mathbf{D}^L, \\
D_j^L &= \overbrace{-\Phi^{-1}(1-\tilde{\alpha})\sqrt{\tau_j(1-\tau_j)} + O(n^{-1/2})}^{\text{from (51)}}.
\end{aligned}
\tag{76}
$$

From (49), $\boldsymbol{\Delta}^L$ is asymptotically multivariate normal with covariance matrix $\underline{\mathcal{V}}$ having elements $\mathcal{V}_{i,k} = \min\{\tau_i,\tau_k\} - \tau_i\tau_k$.

To show $\frac{\partial P}{\partial g_j} = O(1)$,

$$
\begin{aligned}
\frac{\partial P}{\partial g_j} &= \frac{\partial}{\partial g_j}\Phi\left(-\frac{\mu(\mathbf{g},\tilde{\alpha})}{\sigma(\mathbf{g},\tilde{\alpha})}\right) \\
&= \phi(-\mu/\sigma)\left[-\frac{\partial\mu}{\partial g_j}\sigma^{-1} + (-\mu)(-\sigma^{-2})\frac{\partial\sigma}{\partial g_j}\right] \\
&= \overbrace{\phi(-\mu/\sigma)}^{=O(1)}\left[\frac{\mu}{\sigma^2}\frac{\partial\sigma}{\partial g_j} - \frac{1}{\sigma}\frac{\partial\mu}{\partial g_j}\right].
\end{aligned}
$$

Now we must show that σ is uniformly bounded away from zero and that the other terms are uniformly bounded (for large enough n, under Condition $\star$).

For σ being uniformly bounded away from zero,

$$
\sigma^2 = \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j g_i g_j[\min\{\tau_i,\tau_j\} - \tau_i\tau_j], \tag{77}
$$

which is bounded away from zero as long as the g_j are, which they are under Condition $\star$ for large enough n.

For $\mu = O(1)$ uniformly,

$$
\begin{aligned}
\mu &= \sum_{j=1}^{J}\psi_j g_j \overbrace{\left[\big(\mathbb{1}\{\psi_j<0\} - \mathbb{1}\{\psi_j>0\}\big)\Phi^{-1}(1-\tilde{\alpha})\sqrt{\tau_j(1-\tau_j)} + O(n^{-1/2})\right]}^{\text{by (76)}} \\
&= -\Phi^{-1}(1-\tilde{\alpha})\sum_{j=1}^{J}|\psi_j| g_j\sqrt{\tau_j(1-\tau_j)} + O(n^{-1/2}),
\end{aligned}
\tag{78}
$$

which is bounded as long as $\tilde{\alpha}$ is bounded away from zero and the g_j are bounded. By A2, the true $\gamma_j = Q'(\tau_j)$ are indeed bounded away from zero, and under Condition $\star$, the estimated $\widehat{Q'}(\tau_j) \to Q'(\tau_j)$ and thus are also bounded away from zero for large enough n (and since there are a finite number J of these). So, under Condition $\star$, μ is uniformly bounded for large enough n.

For $\frac{\partial\sigma}{\partial g_j} = O(1)$ uniformly,

$$\begin{aligned}\frac{\partial\sqrt{\sigma^2}}{\partial g_j} &= \frac{1}{2\sigma}\frac{\partial\sigma^2}{\partial g_j} \\ &= \frac{1}{2\sigma}\left[\sum_{i\neq j}\psi_i\psi_j g_i[\min\{\tau_i,\tau_j\} - \tau_i\tau_j] + 2\psi_j^2 g_j\tau_j(1-\tau_j)\right] \\ &= O(1)\end{aligned}$$

uniformly since σ is uniformly bounded away from zero (per the above argument) and the g_j are uniformly bounded under Condition $\star$ for large enough n.

For $\frac{\partial\mu}{\partial g_j} = O(1)$ uniformly,

$$\frac{\partial\mu}{\partial g_j} = |\psi_j|\Phi^{-1}(1-\tilde{\alpha})\sqrt{\tau_j(1-\tau_j)} = O(1)$$

uniformly since $\tilde{\alpha} \geq \alpha$, so $\Phi^{-1}(1-\tilde{\alpha}) \leq \Phi^{-1}(1-\alpha) < \infty$.

To show that $\frac{\partial P}{\partial\tilde{\alpha}}$ is (uniformly) bounded away from zero,

$$\begin{aligned}\frac{\partial P}{\partial\tilde{\alpha}} &= \frac{\partial}{\partial\tilde{\alpha}}\Phi\left(-\frac{\mu(\mathbf{g},\tilde{\alpha})}{\sigma(\mathbf{g},\tilde{\alpha})}\right) \\ &= \phi(-\mu/\sigma)\left[-\frac{\partial\mu}{\partial\tilde{\alpha}}\sigma^{-1} + (-\mu)(-\sigma^{-2})\overbrace{\frac{\partial\sigma}{\partial\tilde{\alpha}}}^{=0}\right].\end{aligned}$$

From arguments above, μ is uniformly bounded, and σ is uniformly bounded away from zero, so $-\mu/\sigma$ is uniformly bounded; hence, $\phi(-\mu/\sigma)$ is uniformly bounded away from zero. Also, from (77), σ is uniformly bounded as long as the g_j are, which they are for large enough n under Condition $\star$. Finally, using (78),

$$\frac{\partial\mu}{\partial\tilde{\alpha}} = \frac{1}{\phi\big(\Phi^{-1}(1-\tilde{\alpha})\big)}\sum_{j=1}^{J}|\psi_j|g_j\sqrt{\tau_j(1-\tau_j)} + O(n^{-1/2}).$$

The sum is uniformly bounded away from zero for large enough n under Condition $\star$ since the g_j are. The leading coefficient is, too, since $\phi(\cdot)$ is uniformly (over its argument in $\mathbb{R}$) bounded.

Using mean value expansions where $\tilde{a}$ is between $\tilde{\alpha}\big(\widehat{Q'}(\boldsymbol{\tau})\big)$ and $\tilde{\alpha}\big(Q'(\boldsymbol{\tau})\big)$, and each element of

$\tilde{Q}'(\boldsymbol{\tau})$ is between the corresponding elements of the true and estimated vectors, using (75),

$$\begin{aligned}
\hat{u}_j^h - u_{0,j}^h &\equiv u_j^h\Big(\tilde{\alpha}\big(\widehat{Q'}(\boldsymbol{\tau})\big)\Big) - u_j^h\big(\tilde{\alpha}\big(Q'(\boldsymbol{\tau})\big)\big) \\
&= \Big[\tilde{\alpha}\big(\widehat{Q'}(\boldsymbol{\tau})\big) - \tilde{\alpha}\big(Q'(\boldsymbol{\tau})\big)\Big] u_j^{h\prime}(\tilde{a}) \\
&= \overbrace{\Big[\widehat{Q'}(\boldsymbol{\tau}) - Q'(\boldsymbol{\tau})\Big]'}^{=O(m^{-1/2}\log(n)+m/n)\text{ by (74)}} \overbrace{\tilde{\alpha}'\Big(\tilde{Q}'(\boldsymbol{\tau})\Big)}^{=O(1)\text{ from above}} \overbrace{u_j^{h\prime}(\tilde{a})}^{=O(n^{-1/2})\text{ by (75)}} \\
&= O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big),
\end{aligned} \tag{79}$$

$$Q\Big(\hat{u}_j^h\Big) - Q\Big(u_{0,j}^h\Big) = \Big(\hat{u}_j^h - u_{0,j}^h\Big)Q'(\tilde{u}) = O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big), \tag{80}$$

where $\tilde{u}_j$ is between $\hat{u}_j^h$ and $u_{0,j}^h$ and thus $\tilde{u}_j \to \tau_j$, so for large enough n, $Q'(\tilde{u})$ is uniformly bounded by A2; and similarly with $\hat{u}_j^l$ and $u_{0,j}^l$.

Returning to (45), the term of ultimate interest can be decomposed into

$$\begin{aligned}
E_h &= \mathbb{E}_{\hat{\gamma}}\{\mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)] \mid \hat{\gamma}) - \mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_0^H} > \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\gamma})\} \\
&= \mathbb{E}_{\hat{\gamma}}\{\mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_0^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\gamma}) - \mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)] \mid \hat{\gamma})\} \\
&= \overbrace{\mathbb{E}_{\hat{\gamma}}\{\mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\gamma}) - \mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\hat{\mathbf{u}}^H)] \mid \hat{\gamma})\}}^{E_h^1} \\
&\quad + \overbrace{\mathbb{E}_{\hat{\gamma}}\{\mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_0^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\gamma}) - \mathrm{P}(\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\gamma})\}}^{E_h^2}.
\end{aligned}$$

For E_h^1, GK Lemma 8(ii) is helpful: uniformly over any $\mathbf{u} = \boldsymbol{\tau} + o(1)$, which includes all possible $\hat{\mathbf{u}}^H = \boldsymbol{\tau} + O(n^{-1/2})$, the PDF of $\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H}$ is approximately that of a mean-zero normal distribution with variance $\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H}$:

$$\begin{aligned}
f_{\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H}}(w) &= \phi_{\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H}}(w)\big[1 + O\big(n^{-1/2}[\log(n)]^3\big)\big], \\
\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H} &= \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j\frac{\min\{\hat{u}_i^H, \hat{u}_j^H\} - \hat{u}_i^H\hat{u}_j^H}{f\big(Q(\hat{u}_i^H)\big)f\Big(Q(\hat{u}_j^H)\Big)} \\
&= \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j\Bigg[\frac{\min\{u_{0,i}^H, u_{0,j}^H\} - u_{0,i}^H u_{0,j}^H + \overbrace{O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)}^{(79)}}{f\Big(Q(u_{0,i}^H)\Big)f\Big(Q(u_{0,j}^H)\Big)} \\
&\qquad + \overbrace{O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)}^{(79),\text{ A2}}\Bigg] \\
&= \overbrace{\sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j\frac{\min\{u_{0,i}^H, u_{0,j}^H\} - u_{0,i}^H u_{0,j}^H}{f\Big(Q(u_{0,i}^H)\Big)f\Big(Q(u_{0,j}^H)\Big)}}^{\equiv\mathcal{V}_{\boldsymbol{\psi}}^{0,H}} + O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)
\end{aligned}$$

$$= \mathcal{V}_{\boldsymbol{\psi}}^{0,H} + O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big),$$

using (for the denominator) the mean value expansion $Q'(\hat{u}_j^H) = Q'(u_{0,j}^H) + Q''(\tilde{u}_j)(\hat{u}_j^H - u_{0,j}^H)$, where $\tilde{u}_j$ is between $\hat{u}_j^H$ and $u_{0,j}^H$ and thus converges to τ_j, so for large enough n, A2 ensures $Q''(\tilde{u}_j)$ is uniformly $O(1)$. In terms of the standard normal PDF $\phi(\cdot)$, a mean-zero normal PDF with variance σ^2 can be written $\sigma^{-1}\phi(\cdot/\sigma)$. Thus, in terms of the reference PDF with $\mathbf{u}_0^H$,

$$
\begin{aligned}
\phi_{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}(w) &= \sigma_0^{-1}\phi(w/\sigma_0),\\
\phi_{\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H}}(w) &= \hat{\sigma}^{-1}\phi(w/\hat{\sigma}) = (\sigma_0/\hat{\sigma})\phi_{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}(w\sigma_0/\hat{\sigma}),\\
(\sigma_0/\hat{\sigma}) &\equiv \sqrt{\frac{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}{\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H}}} = \sqrt{\frac{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}{\mathcal{V}_{\boldsymbol{\psi}}^{0,H} + O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)}}\\
&= \sqrt{1 + O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)}\\
&= 1 + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big).
\end{aligned}
$$

Since the points of evaluation of interest w are uniformly $O(1)$,

$$w\Big[1 + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big)\Big] = w + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big),$$

so altogether,

$$
\begin{aligned}
f_{\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H}}(w) &= \phi_{\mathcal{V}_{\boldsymbol{\psi}}^{\hat{\mathbf{u}}^H}}(w)\Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big]\\
&= (\sigma_0/\hat{\sigma})\phi_{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}(w\sigma_0/\hat{\sigma})\Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big]\\
&= \Big[1 + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big)\Big]\\
&\quad \times \phi_{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}\Big(w + O\big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\big)\Big)\\
&\quad \times \Big[1 + O\big(n^{-1/2}[\log(n)]^3\big)\Big]\\
&= \phi_{\mathcal{V}_{\boldsymbol{\psi}}^{0,H}}(w)\Big[1 + O\big(n^{-1/2}[\log(n)]^3\big)\Big]
\end{aligned}
\tag{81}
$$

since $m \to \infty$ implies $m^{-1/2}n^{-1/2} = o(n^{-1/2})$ and $m = o(n)$ implies $mn^{-3/2} = o(n^{-1/2})$. The error inside the argument to the PDF can be pulled out multiplicatively as in (81) because w is $O(1)$ and the variance $\mathcal{V}_{\boldsymbol{\psi}}^{0,H}$ is fixed, so

$$
\begin{aligned}
&\exp\Big\{-\Big[w + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big)\Big]^2/(2\mathcal{V}_{\boldsymbol{\psi}}^{0,H})\Big\}\\
&= \exp\Big\{-w^2/(2\mathcal{V}_{\boldsymbol{\psi}}^{0,H}) + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big)\Big\}\\
&= \exp\Big\{-w^2/(2\mathcal{V}_{\boldsymbol{\psi}}^{0,H})\Big\}\overbrace{\exp\Big\{O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big)\Big\}}^{=1+O\left(m^{-1/2}n^{-1/2}\log(n)+mn^{-3/2}\right)}.
\end{aligned}
\tag{82}
$$

(Alternatively, one can use the MVT and the fact that the normal PDF derivative is proportional to the PDF.) So, the PDF evaluated over a set of uniformly bounded values is uniformly approximated over all possible $\hat{\mathbf{u}}^H$ by that of a mean-zero normal with variance $\mathcal{V}^{0,H}_{\boldsymbol{\psi}}$, up to a multiplicative error.

Given a value of $\hat{\mathbf{u}}^H$, the mean value theorem gives

$$
\begin{aligned}
E^1_h(\hat{\mathbf{u}}^H) &= \int_{\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\hat{\mathbf{u}}^H)]}^{\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)]} f_{\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}}}(w)\,dw \\
&= \Big\{\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)] - \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau})-Q(\hat{\mathbf{u}}^H)]\Big\} f_{\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\boldsymbol{\Lambda}}}(\tilde{w}) \\
&= \sqrt{n}\boldsymbol{\psi}'\Big[Q(\hat{\mathbf{u}}^H)-Q(\mathbf{u}^H_0)\Big] \overbrace{\phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}(\tilde{w})\Big[1+O\Big(n^{-1/2}[\log(n)]^3\Big)\Big]}^{\text{by (81)}} \\
&= \sqrt{n}\boldsymbol{\psi}' \overbrace{\Big[\widehat{Q'}(\boldsymbol{\tau})-Q'(\boldsymbol{\tau})\Big]O(1)O(n^{-1/2})O(1)}^{\text{by (79) and (80)}} \phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}(\tilde{w})\Big[1+O\Big(n^{-1/2}[\log(n)]^3\Big)\Big] \\
&= \boldsymbol{\psi}'\Big[\widehat{Q'}(\boldsymbol{\tau})-Q'(\boldsymbol{\tau})\Big]\phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}(\tilde{w})\Big[1+O\big(n^{-1/2}[\log(n)]^3\big)\Big] \qquad (83)
\end{aligned}
$$

where $\tilde{w}$ is between the limits of integration and thus $O(1)$. Actually, $\tilde{w}$ is pinned down much more precisely: using (80),

$$
\begin{aligned}
\tilde{w} &= \sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big] + O\big(\sqrt{n}\boldsymbol{\psi}' \overbrace{\Big[Q(\hat{\mathbf{u}}^H)-Q(\mathbf{u}^H_0)\Big]}^{=O\left(m^{-1/2}n^{-1/2}\log(n)+mn^{-3/2}\right)\text{ by (80)}} \big) \\
&= \overbrace{\sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big]}^{=O(1)} + O\big(m^{-1/2}\log(n)+mn^{-1}\big).
\end{aligned}
$$

By the same arguments leading to (82), this error becomes multiplicative when pulling it out of the normal PDF:

$$
\begin{aligned}
\phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}(\tilde{w}) &= \phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}\Big(\sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big] + O\Big(m^{-1/2}\log(n)+mn^{-1}\Big)\Big) \\
&= \phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}\big(\sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big]\big)\Big[1+O\Big(m^{-1/2}\log(n)+mn^{-1}\Big)\Big].
\end{aligned}
$$

So, up to a uniform multiplicative error, the only random variables left in E^1_h are the sparsity estimators, $\widehat{Q'}(\tau_j)$. Altogether,

$$
\begin{aligned}
E^1_h &= \mathbb{E}\Bigg\{\boldsymbol{\psi}'\Big[\widehat{Q'}(\boldsymbol{\tau})-Q'(\boldsymbol{\tau})\Big]\phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}\big(\sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big]\big) \\
&\qquad \times \overbrace{\Big[1+O\Big(m^{-1/2}\log(n)+mn^{-1}\Big)\Big]\Big[1+O\Big(n^{-1/2}[\log(n)]^3\Big)\Big]}^{\text{uniform over }\hat{\mathbf{u}}^H}\Bigg\} \\
&= \overbrace{\boldsymbol{\psi}'}^{=O(1)}\ \mathbb{E}\Big\{\Big[1+O\Big(m^{-1/2}\log(n)+mn^{-1}\Big)\Big]\Big[\widehat{Q'}(\boldsymbol{\tau})-Q'(\boldsymbol{\tau})\Big]\Big\} \overbrace{\phi_{\mathcal{V}^{0,H}_{\boldsymbol{\psi}}}\big(\sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau})-Q(\mathbf{u}^H_0)\big]\big)}^{=O(1)} \\
&= O(\mathbb{E}(AB)), \qquad (84) \\
A &= 1+O\Big(m^{-1/2}\log(n)+mn^{-1}\Big),
\end{aligned}
$$

$$B = \max_j \Big\{ \widehat{Q'}(\tau_j) - Q'(\tau_j) \Big\}.$$

Now,

$$\begin{aligned}
\mathbb{E}(AB) &= \mathrm{Cov}(A,B) + \mathbb{E}(A)\,\mathbb{E}(B), \\
|\mathrm{Cov}(A,B)| &= \Big|\mathrm{Corr}(A,B)\sqrt{\mathrm{Var}(A)\,\mathrm{Var}(B)}\Big| \le \sqrt{\mathrm{Var}(A)\,\mathrm{Var}(B)}.
\end{aligned}$$

From equations (2.5) and (2.6) in Bloch and Gastwirth (1968, p. 1084),

$$\mathbb{E}(B) = O\big(m^2/n^2\big), \quad \mathrm{Var}(B) = O\big(m^{-1}\big).$$

Also,

$$\begin{aligned}
\mathbb{E}(A) &= \mathbb{E}\Big[1 + O\Big(m^{-1/2}\log(n) + mn^{-1}\Big)\Big] = O(1), \\
\mathrm{Var}(A) &\le \Big\{1 + O\Big(m^{-1/2}\log(n) + mn^{-1}\Big) - \Big[1 + O\Big(m^{-1/2}\log(n) + mn^{-1}\Big)\Big]\Big\}^2 \\
&= O\big(m^{-1}[\log(n)]^2 + m^2 n^{-2}\big), \\
|\mathbb{E}(AB)| &\le \sqrt{O(m^{-1}[\log(n)]^2 + m^2n^{-2})O(m^{-1})} + O(1)O(m^2/n^2) \\
&= O\big(m^{-1}\log(n) + m^2/n^2\big).
\end{aligned}$$

Using $m \asymp n^{2/3}$ attains the (nearly) minimum rate of $O\big(n^{-2/3}\log(n)\big)$. This can be slightly improved to $O\big(n^{-2/3}[\log(n)]^{2/3}\big)$ with $m \asymp n^{2/3}[\log(n)]^{1/3}$, but the practical difference is negligible, so we prefer $m \asymp n^{2/3}$ for simplicity. With any $n^{1/2} \lesssim m \lesssim n^{3/4}$, the rate is no greater than $T_h = O(n^{-1/2}\log(n))$.

Finally, for E_h^2, recall that for any $\mathbf{t} = \boldsymbol{\tau} + o(1)$, $\mathbb{W}^{\mathbf{t}}_{\mathbf{C},\mathbf{\Lambda}}$ has (asymptotically) a mean-zero normal distribution with variance $\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}$, so its CDF can be written in terms of the standard normal CDF $\Phi(\cdot)$, as $\Phi\Big(\cdot/\sqrt{\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}}\Big)$. Denote the fixed (wrt $\hat{\mathbf{u}}^H$), $O(1)$ point of evaluation $w = \sqrt{n}\boldsymbol{\psi}'\big[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)\big]$. Then,

$$\frac{\partial \Phi\Big(w/\sqrt{\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}}\Big)}{\partial \mathbf{t}} = \phi\Big(w/\sqrt{\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}}\Big)\frac{\partial w(\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}})^{-1/2}}{\partial \mathbf{t}} = \phi\Big(w/\sqrt{\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}}\Big)\frac{-w}{2(\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}})^{3/2}}\frac{\partial \mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}}{\partial \mathbf{t}} = O(1) \tag{85}$$

since all three terms in the product are $O(1)$.

Altogether,

$$\begin{aligned}
|E_h^2| &= \Big|\mathbb{E}_{\hat{\boldsymbol{\gamma}}}\{\mathrm{P}(\mathbb{W}^{\mathbf{u}_0^H}_{\mathbf{C},\mathbf{\Lambda}} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}}) - \mathrm{P}(\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\mathbf{\Lambda}} < \sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}})\}\Big| \\
&= \Big|\mathbb{E}_{\hat{\boldsymbol{\gamma}}}\Big\{F_{\mathbb{W}^{\mathbf{u}_0^H}_{\mathbf{C},\mathbf{\Lambda}}|\hat{\boldsymbol{\gamma}}}\big(\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}}\big) - F_{\mathbb{W}^{\hat{\mathbf{u}}^H}_{\mathbf{C},\mathbf{\Lambda}}|\hat{\boldsymbol{\gamma}}}\big(\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}}\big)\Big\}\Big| \\
&= \Bigg|\mathbb{E}_{\hat{\boldsymbol{\gamma}}}\Bigg\{\overbrace{(\mathbf{u}_0^H - \hat{\mathbf{u}}^H)'}^{=O\left(m^{-1/2}n^{-1/2}\log(n)+mn^{-3/2}\right)\text{ by (79)}}\overbrace{\frac{dF_{\mathbb{W}^{\mathbf{t}}_{\mathbf{C},\mathbf{\Lambda}}|\hat{\boldsymbol{\gamma}}}(\sqrt{n}\boldsymbol{\psi}'[Q(\boldsymbol{\tau}) - Q(\mathbf{u}_0^H)] \mid \hat{\boldsymbol{\gamma}})}{d\mathbf{t}}\bigg|_{\mathbf{t}=\tilde{\mathbf{u}}}}^{=O(1)\text{ by (85)}}\Bigg\}\Bigg|
\end{aligned}$$

$$= O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big), \tag{86}$$

where $\tilde{\mathbf{u}}$ is between $\mathbf{u}_0^H$ and $\hat{\mathbf{u}}^H$. The rate in (86) is even smaller than necessary. For the two-sided case, we only need $E_h^2 = O(n^{-2/3})$, but with $m \asymp n^{2/3}$ we get $O\big(n^{-5/6}\log(n)\big)$. For the one-sided case, we need $E_h^2 = O\big(n^{-1/2}\log(n)\big)$, but over $m \in [n^{1/2}, n^{3/4}]$ the maximum of (86) is $O\big(n^{-3/4}\log(n)\big)$.

The foregoing arguments and rates are all the same for E_l. □

A.4 Theorem for CI for difference of linear combination of quantiles (and QD)

We first state (and then prove) Theorem 10, where the object of interest is

$$D = \sum_{j=1}^{J} \psi_j [Q_Y(\tau_j) - Q_X(\tau_j)].$$

In the special case $J = 1$, D is the τ-QD. If $\boldsymbol{\psi} = (-1, 1)'$, then D is a difference of IQRs.

For a lower one-sided CI, using (7) and (34), $\tilde{\alpha}$ satisfies

$$1 - \alpha = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \left\{\widehat{Q'_Y}(\tau_j)\Big[\tilde{Q}^I_{U_y}\big(u^H_{y,j}(\tilde{\alpha})\big) - \tau_j\Big] - \widehat{Q'_X}(\tau_j)\Big[\tilde{Q}^I_{U_x}\big(u^L_{x,j}(\tilde{\alpha})\big) - \tau_j\Big]\right\} > 0\right). \tag{87}$$

The $1 - \alpha$ CI is then

$$\left(-\infty, \sum_{j=1}^{J} \psi_j \Big[\hat{Q}^L_Y\big(u^H_{y,j}(\tilde{\alpha})\big) - \hat{Q}^L_X\big(u^L_{x,j}(\tilde{\alpha})\big)\Big]\right). \tag{88}$$

For an upper one-sided CI, the analogs of (87) and (88) are

$$1 - \alpha = \mathrm{P}\left(\sum_{j=1}^{J} \psi_j \left\{\widehat{Q'_Y}(\tau_j)\Big[\tilde{Q}^I_{U_y}\big(u^L_{y,j}(\tilde{\alpha})\big) - \tau_j\Big] - \widehat{Q'_X}(\tau_j)\Big[\tilde{Q}^I_{U_x}\big(u^H_{x,j}(\tilde{\alpha})\big) - \tau_j\Big]\right\} < 0\right), \tag{89}$$

$$\left(\sum_{j=1}^{J} \psi_j \Big[\hat{Q}^L_Y\big(u^L_{y,j}(\tilde{\alpha})\big) - \hat{Q}^L_X\big(u^H_{x,j}(\tilde{\alpha})\big)\Big], \infty\right). \tag{90}$$

Theorem 10. *Let Assumptions A1 and A2 hold.*

(i) The one-sided CIs in (88) *and* (90) *both have CPE of order* $O\big(n^{-1/2}\log(n)\big)$ *if all* $\widehat{Q'_X}(\tau_j)$ *and* $\widehat{Q'_Y}(\tau_j)$ *are estimated by* (12) *with smoothing parameters* $m_{x,j}$ *and* $m_{y,j}$ *having rates larger than* $n^{1/2}$ *and smaller than* $n^{3/4}$.

(ii) Two-sided CIs, formed by the intersection of upper and lower one-sided $1-\alpha/2$ *CIs, have CPE of order* $O\big(n^{-2/3}\log(n)\big)$ *if all* $\widehat{Q'_X}(\tau_j)$ *and* $\widehat{Q'_Y}(\tau_j)$ *are estimated by* (12) *with* $m_{x,j} \asymp n^{2/3}$ *and* $m_{y,j} \asymp n^{2/3}$.

(iii) The asymptotic probabilities of excluding $D_n = \boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_X(\boldsymbol{\tau}) + \boldsymbol{\kappa} n_y^{-1/2}\Big]$ *from lower one-sided (l), upper one-sided (u), and equal-tailed two-sided (t) CIs (i.e., asymptotic power*

of the corresponding hypothesis tests) are

$$\mathcal{P}_n^l(D_n) \to \Phi(z_\alpha + S),\ \ \mathcal{P}_n^u(D_n) \to \Phi(z_\alpha - S),\ \ \mathcal{P}_n^t(D_n) \to \Phi\big(z_{\alpha/2} + S\big) + \Phi\big(z_{\alpha/2} - S\big),$$

where $S \equiv \boldsymbol{\psi}'\boldsymbol{\kappa}/\sqrt{\mathcal{V}_{\boldsymbol{\psi},x} + \delta^2\mathcal{V}_{\boldsymbol{\psi},y}}$*, and* $\mathcal{V}_{\boldsymbol{\psi},x}$ *and* $\mathcal{V}_{\boldsymbol{\psi},y}$ *are from* (38) *for the* X *and* Y *population distributions, respectively.*

Proof. Let

$$\begin{aligned}
\hat{\mathbf{u}}_y^H &= \{u_{y,j}^H(\tilde{\alpha}[\hat{\boldsymbol{\gamma}}_x, \hat{\boldsymbol{\gamma}}_y])\}_{j=1}^J, \quad \mathbf{u}_{0,y}^H = \{u_{y,j}^H(\tilde{\alpha}[\boldsymbol{\gamma}_x, \boldsymbol{\gamma}_y])\}_{j=1}^J, \\
\hat{\mathbf{u}}_x^H &= \{u_{x,j}^L(\tilde{\alpha}[\hat{\boldsymbol{\gamma}}_x, \hat{\boldsymbol{\gamma}}_y])\}_{j=1}^J, \quad \mathbf{u}_{0,x}^H = \{u_{x,j}^L(\tilde{\alpha}[\boldsymbol{\gamma}_x, \boldsymbol{\gamma}_y])\}_{j=1}^J.
\end{aligned}$$

From A1, $\sqrt{n_x/n_y} = \delta + O(n^{-1})$ with $0 < \delta < \infty$. Referring back to (88), and with steps similar to the derivation of (43), true CP is

$$\begin{aligned}
&\mathrm{P}\Big(\boldsymbol{\psi}'\Big[\hat{Q}_Y^L(\hat{\mathbf{u}}_y^H) - \hat{Q}_X^L(\hat{\mathbf{u}}_x^H)\Big] > \boldsymbol{\psi}'[Q_Y(\boldsymbol{\tau}) - Q_X(\boldsymbol{\tau})]\Big) \\
&= \mathrm{P}\Big(\boldsymbol{\psi}'\Big[\tilde{Q}_Y^I(\hat{\mathbf{u}}_y^H) - \tilde{Q}_X^I(\hat{\mathbf{u}}_x^H)\Big] > \boldsymbol{\psi}'[Q_Y(\boldsymbol{\tau}) - Q_X(\boldsymbol{\tau})]\Big) + \overbrace{O(n^{-1})}^{\text{by Theorem 1}} \\
&= \mathrm{P}\Bigg(\overbrace{\sqrt{\frac{n_x}{n_y}}}^{=\delta+O(n^{-1})}\sqrt{n_y}\boldsymbol{\psi}'\Big[\tilde{Q}_Y^I(\hat{\mathbf{u}}_y^H) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[\tilde{Q}_X^I(\hat{\mathbf{u}}_x^H) - Q_X(\hat{\mathbf{u}}_x^H)\Big] \\
&\qquad > \overbrace{\sqrt{\frac{n_x}{n_y}}}^{=\delta+O(n^{-1})}\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) + O(n^{-1}) \\
&= \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H} \\
&\qquad > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) \\
&\quad + \Bigg[\mathrm{P}\Bigg(\delta\sqrt{n_y}\boldsymbol{\psi}'\Big[\tilde{Q}_Y^I(\hat{\mathbf{u}}_y^H) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[\tilde{Q}_X^I(\hat{\mathbf{u}}_x^H) - Q_X(\hat{\mathbf{u}}_x^H)\Big] \\
&\qquad\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) \\
&\qquad\quad - \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H} \\
&\qquad\qquad > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg)\Bigg] \\
&\quad + O(n^{-1}) \\
&= \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H}
\end{aligned}$$

$$
\begin{aligned}
&> \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) \\
&+ \overbrace{O\Big(n^{-3/2}[\log(n)]^3\Big)}^{\text{by (42) and independence (A1)}} + O(n^{-1}) \\
&= \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H} \\
&\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) \\
&+ O(n^{-1}) \\
&= \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,y}^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,x}^H} \\
&\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big]\Bigg) \\
&+ E_{h,2} + O(n^{-1}) \\
&= \overbrace{\mathrm{P}\Bigg(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{0,y,j}^H\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{0,x,j}^H\big) - \tau_j\Big]\Big\} > 0\Bigg)}^{=1-\alpha \text{ by (87)}} \\
&+ T_{h,2} + E_{h,2} + O(n^{-1}), \qquad (91) \\
E_{h,2} &\equiv \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H} \\
&\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\Bigg) \\
&- \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,y}^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,x}^H} \\
&\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big]\Bigg), \\
T_{h,2} &\equiv \mathrm{P}\Bigg(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,y}^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,x}^H} \\
&\quad > \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big]\Bigg) \\
&- \mathrm{P}\Bigg(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{0,y,j}^H\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{0,x,j}^H\big) - \tau_j\Big]\Big\} > 0\Bigg) \\
&= \mathrm{P}\Bigg(\delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y\Big(\tilde{Q}_{U_y}^I(\mathbf{u}_{0,y}^H)\Big) - Q_Y(\mathbf{u}_{0,y}^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X\Big(\tilde{Q}_{U_x}^I(\mathbf{u}_{0,x}^H)\Big) - Q_X(\mathbf{u}_{0,x}^H)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&> \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big]\Bigg) \\
&+ \quad \overbrace{O\Big(n^{-3/2}[\log(n)]^3\Big)}^{\text{by (42) and independence (A1)}} \\
&- \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{0,y,j}^H\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{0,x,j}^H\big) - \tau_j\Big]\Big\} > 0\right) \\
= {}& \mathrm{P}\Bigg(\delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y\Big(\tilde{Q}_{U_y}^I(\mathbf{u}_{0,y}^H)\Big) - Q_Y(\boldsymbol{\tau})\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X\Big(\tilde{Q}_{U_x}^I(\mathbf{u}_{0,x}^H)\Big) - Q_X(\boldsymbol{\tau})\Big] \\
&> 0\Bigg) \\
&- \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{0,y,j}^H\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{0,x,j}^H\big) - \tau_j\Big]\Big\} > 0\right) \\
&+ O\Big(n^{-3/2}[\log(n)]^3\Big) \\
= {}& \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big\{\Big[Q_Y\Big(\tilde{Q}_{U_y}^I(u_{0,y,j}^H)\Big) - Q_Y(\tau_j)\Big] - \Big[Q_X\Big(\tilde{Q}_{U_x}^I(u_{0,x,j}^H)\Big) - Q_X(\tau_j)\Big]\Big\} > 0\right) \\
&- \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{0,y,j}^H\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{0,x,j}^H\big) - \tau_j\Big]\Big\} > 0\right) \\
&+ O\big(n^{-3/2}[\log(n)]^3 + n^{-1}\big),
\end{aligned}
$$

where the final n^{-1} in the remainder could be avoided by rewriting everything with $\sqrt{n_x/n_y}$ instead of introducing δ, but it is smaller-order anyway. Similar to before, the term $T_{h,2}$ is the CPE induced by the linearizations of the quantile functions, and $E_{h,2}$ is CPE induced by estimation error of the nuisance parameters. The upper one-sided derivation yields similar terms, denoted $T_{l,2}$ and $E_{l,2}$. The derivations of the orders of magnitude of $T_{h,2}$ and $E_{h,2}$ parallel those of T_h and E_h in the proof of Theorem 7.

The proof of part (i) follows by applying Lemmas 11 and 12, which respectively have $T_{h,2} = O\big(n^{-1/2}\log(n)\big)$ and $E_{h,2} = O\big(m^{-1}\log(n) + (m/n)^2\big)$ for common smoothing parameter rate m and common sample size rate n (as in A1), and similarly for $T_{l,2}$ and $E_{l,2}$. As long as $n^{1/2} \lesssim m \lesssim n^{3/4}$, the dominant CPE term is order $O\big(n^{-1/2}\log(n)\big)$.

The proof of part (ii) also follows by applying Lemmas 11 and 12, which additionally give $T_{h,2} + T_{l,2} = O\big(n^{-1}\log(n)\big)$. Thus, CPE is $O\big(n^{-1}\log(n)\big) + O\big(m^{-1}\log(n) + (m/n)^2\big)$. Now, the second term dominates, and it is minimized by $m \asymp n^{2/3}$, leaving CPE of order $O\big(n^{-2/3}\log(n)\big)$.

The proof of part (iii) remains. It parallels the proof of Theorem 7(iii). The addition of Y variables provides little complication since the samples are assumed independent, so the asymptotic normal distributions from GK Lemma 8 are independent, which implies their sum is normal with variance equal to the sum of variances. Also, the sample size ratio δ^2 must be incorporated. Otherwise, the steps are identical.

One-sided power against

$$H_0 : D_n = \boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_X(\boldsymbol{\tau}) + \boldsymbol{\kappa} n^{-1/2}\Big]$$

with $\boldsymbol{\psi}'\boldsymbol{\kappa} > 0$ is the probability that D_n is not contained in the lower one-sided CI. Below, $\tilde{u}_{y,j}$ comes from the mean value theorem and lies between τ_j and $u^H_{y,j}$. Since $u^H_{y,j} \to \tau_j$ by Lemma 6, $\tilde{u}_{y,j} \to \tau_j$, so for large enough n, all $\tilde{u}_{y,j}$ lie within an arbitrarily small neighborhood of τ_j and thus A2 uniformly bounds $Q''_Y(\tilde{u}_{y,j}) = O(1)$. The same argument applies to $Q''_X(\tilde{u}_{x,j}) = O(1)$. The CI exclusion probability is

$$\begin{aligned}
&\mathcal{P}^l_n(D_n) \\
&= \mathrm{P}\left\{\sum_{j=1}^{J} \psi_j \Big[\hat{Q}^L_Y(u^H_{y,j}(\tilde{\alpha}_j)) - Q_Y(\tau_j) - \Big[\hat{Q}^L_X(u^L_{x,j}(\tilde{\alpha}_j)) - Q_X(\tau_j)\Big]\Big] < n^{-1/2}\boldsymbol{\psi}'\boldsymbol{\kappa}\right\} \\
&= \mathrm{P}\Big\{\boldsymbol{\psi}'\Big[\hat{Q}^L_Y\big(\mathbf{u}^H_y(\tilde{\boldsymbol{\alpha}})\big) - Q_Y\big(\mathbf{u}^H_y(\tilde{\boldsymbol{\alpha}})\big)\Big] - \boldsymbol{\psi}'\Big[\hat{Q}^L_X\big(\mathbf{u}^L_x(\tilde{\boldsymbol{\alpha}})\big) - Q_X\big(\mathbf{u}^L_x(\tilde{\boldsymbol{\alpha}})\big)\Big] \\
&\qquad\qquad < n^{-1/2}\boldsymbol{\psi}'\boldsymbol{\kappa} - \boldsymbol{\psi}'\big[Q_Y\big(\mathbf{u}^H_y(\tilde{\boldsymbol{\alpha}})\big) - Q_Y(\boldsymbol{\tau})\big] - \boldsymbol{\psi}'\big[Q_X\big(\mathbf{u}^L_x(\tilde{\boldsymbol{\alpha}})\big) - Q_X(\boldsymbol{\tau})\big]\Big\} \\
&= \mathrm{P}\Big\{\delta\sqrt{n_y}\boldsymbol{\psi}'\Big[\hat{Q}^L_Y\big(\mathbf{u}^H_y(\tilde{\boldsymbol{\alpha}})\big) - Q_Y\big(\mathbf{u}^H_y(\tilde{\boldsymbol{\alpha}})\big)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[\hat{Q}^L_X\big(\mathbf{u}^L_x(\tilde{\boldsymbol{\alpha}})\big) - Q_X\big(\mathbf{u}^L_x(\tilde{\boldsymbol{\alpha}})\big)\Big] \\
&\qquad < \boldsymbol{\psi}'\boldsymbol{\kappa} - \delta\sqrt{n_y}\sum_{j=1}^{J}\psi_j\Big[Q'_Y(\tau_j)(u^H_{y,j} - \tau_j) + (1/2)\overbrace{Q''_Y(\tilde{u}_j)}^{=O(1)}\overbrace{(u^H_{y,j} - \tau_j)^2}^{=O(n^{-1})\text{ by Lemma 6}}\Big] \\
&\qquad\quad - \sqrt{n_x}\sum_{j=1}^{J}\psi_j\Big[Q'_X(\tau_j)(u^L_{x,j} - \tau_j) + (1/2)\overbrace{Q''_X(\tilde{u}_{x,j})}^{=O(1)}\overbrace{(u^L_{x,j} - \tau_j)^2}^{=O(n^{-1})\text{ by Lemma 6}}\Big]\Big\} \\
&= \overbrace{\Phi\left(\frac{\sum_{j=1}^{J}\psi_j\kappa_j - \overbrace{\psi_j Q'_Y(\tau_j)\delta}^{=O(1)}\overbrace{\sqrt{n_y}\big[u^H_{y,j}(\tilde{\alpha}_j) - \tau_j\big]}^{\text{apply Lemma 6}} - \overbrace{\psi_j Q'_X(\tau_j)}^{=O(1)}\overbrace{\sqrt{n_x}\big[u^L_{x,j}(\tilde{\alpha}_j) - \tau_j\big]}^{\text{apply Lemma 6}} + O(n^{-1/2})}{\sqrt{\hat{\mathcal{V}}_{\boldsymbol{\psi},x} + \delta^2\hat{\mathcal{V}}_{\boldsymbol{\psi},y}}}\right)}^{\text{by GK Lemma 8 and independence (A1)}} \\
&\quad + O\Big(n^{-1/2}[\log(n)]^3\Big) \\
&= \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}} + \delta^2\mathcal{V}_{\boldsymbol{\psi},y}}}\right. \\
&\qquad \left. - \overbrace{\frac{1}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}} + \delta^2\mathcal{V}_{\boldsymbol{\psi},y}}}\sum_{j=1}^{J}\psi_j z_{1-\tilde{\alpha}_j}\sqrt{\tau_j(1-\tau_j)}\big[\delta Q'_Y(\tau_j) + Q'_X(\tau_j)\big]}^{\to z_{1-\alpha}\text{ to control size when }\boldsymbol{\kappa}=\mathbf{0}} + O(n^{-1/2})\right) \\
&\quad + O\Big(n^{-1/2}[\log(n)]^3\Big)
\end{aligned}$$

$$\to \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}}+\delta^2\mathcal{V}_{\boldsymbol{\psi},y}}} - z_{1-\alpha}\right) = \Phi\left(\frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}}+\delta^2\mathcal{V}_{\boldsymbol{\psi},y}}} + z_{\alpha}\right),$$

where

$$\hat{\mathcal{V}}_{\boldsymbol{\psi},y} \equiv \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j \frac{\min\left\{u^H_{y,i}(\tilde{\alpha}_i), u^H_{y,j}(\tilde{\alpha}_j)\right\} - u^H_{y,i}(\tilde{\alpha}_i)u^H_{y,j}(\tilde{\alpha}_j)}{f_Y\left(Q_Y(u^H_{y,i}(\tilde{\alpha}_i))\right)f_Y\left(Q_Y(u^H_{y,j}(\tilde{\alpha}_j))\right)} \to \mathcal{V}_{\boldsymbol{\psi},y},$$

$$\hat{\mathcal{V}}_{\boldsymbol{\psi},x} \equiv \sum_{i=1}^{J}\sum_{j=1}^{J}\psi_i\psi_j \frac{\min\left\{u^L_{x,i}(\tilde{\alpha}_i), u^L_{x,j}(\tilde{\alpha}_j)\right\} - u^L_{x,i}(\tilde{\alpha}_i)u^L_{x,j}(\tilde{\alpha}_j)}{f_X\left(Q_X(u^L_{x,i}(\tilde{\alpha}_i))\right)f_X\left(Q_X(u^L_{x,j}(\tilde{\alpha}_j))\right)} \to \mathcal{V}_{\boldsymbol{\psi},x},$$

$$\sqrt{\hat{\mathcal{V}}_{\boldsymbol{\psi},x}+\delta^2\hat{\mathcal{V}}_{\boldsymbol{\psi},y}} \to \sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}}+\delta^2\mathcal{V}_{\boldsymbol{\psi},y}},$$

applying the continuous mapping theorem in the final line.

The upper one-sided case follows similarly.

For the two-sided case, since the two-sided CI is the intersection of the upper and lower one-sided $1-\alpha/2$ CIs, the exclusion probability is

$$\begin{aligned}\mathcal{P}^t_n(D_n,\alpha) &= \mathcal{P}^u_n(D_n,\alpha/2) + \mathcal{P}^l_n(D_n,\alpha/2) \\ &\to \Phi\left(z_{\alpha/2} + \frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}}+\delta^2\mathcal{V}_{\boldsymbol{\psi},y}}}\right) + \Phi\left(z_{\alpha/2} - \frac{\boldsymbol{\psi}'\boldsymbol{\kappa}}{\sqrt{\mathcal{V}_{\boldsymbol{\psi},\mathbf{x}}+\delta^2\mathcal{V}_{\boldsymbol{\psi},y}}}\right).\end{aligned}$$ □

A.4.1 CPE from two-sample Taylor approximations: $T_{h,2}$, $T_{l,2}$

Lemma 11. *Under the assumptions of Theorem 10, the term $T_{h,2}$ from* (91) *is of order $T_{h,2} = O(n^{-1/2}\log(n))$, and similarly $T_{l,2} = O(n^{-1/2}\log(n))$ for the corresponding upper one-sided term. Additionally, $T_{h,2}+T_{l,2} = O(n^{-1}[\log(n)]^2)$.*

Proof. The proof largely parallels that of Lemma 8; here, we point out non-trivial differences. Because the samples are assumed independent in A1, joint PDFs of objects involving both samples are simply the product of the marginal PDFs. For example, in the proof of Lemma 8, the PDFs of $\boldsymbol{\Delta}^H$ and $\boldsymbol{\Delta}^L$ were given. Continuing to use subscripts to denote the sample (x or y), since $\boldsymbol{\Delta}^H_y \perp\!\!\!\perp \boldsymbol{\Delta}^L_x$ by A1, their joint PDF is the product of their PDFs. Also, the conditions in A2 for both population distributions are the same as for the population distribution in the proof of Lemma 8, so remainder terms have the same bounds as before.

Applying the same arguments as before, $T_{h,2}(\tilde{\alpha})$ can be decomposed into

$$\begin{aligned}T_{h,2}(\tilde{\alpha}) &= T_{H,2,1} - T_{H,2,2} + O(n^{-1}[\log(n)]^{3/2}), \\ T_{H,2,2} &\equiv \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\left\{Q'_Y(\tau_j)\left[\tilde{Q}^I_{U_y}\left(u^H_{y,j}(\tilde{\alpha})\right)-\tau_j\right] - Q'_X(\tau_j)\left[\tilde{Q}^I_{U_x}\left(u^L_{x,j}(\tilde{\alpha})\right)-\tau_j\right]\right\} > 0\right) \\ &= \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\left\{\delta Q'_Y(\tau_j)\left(\Delta^H_{y,j}+D^H_{y,j}\right) - Q'_X(\tau_j)\left(\Delta^L_{x,j}+D^L_{x,j}\right)\right\} > 0\right), \\ T_{H,2,1} &\equiv \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\left\{\delta Q'_Y(\tau_j)\left(\Delta^H_{y,j}+D^H_{y,j}\right) - Q'_X(\tau_j)\left(\Delta^L_{x,j}+D^L_{x,j}\right)\right\}\right.\end{aligned}$$

$$> -\frac{n_x^{-1/2}}{2}\sum_{j=1}^{J}\psi_j\Big[\delta^2 Q_Y''(\tau_j)\big(\Delta_{y,j}^H + D_{y,j}^H\big)^2 - Q_X''(\tau_j)\big(\Delta_{x,j}^L + D_{x,j}^L\big)^2\Big]\Bigg),$$

$$\Delta_{y,j}^H \equiv \sqrt{n_y}\Big[\tilde{Q}_{U_y}^I\big(u_{y,j}^H(\tilde\alpha)\big) - u_{y,j}^H(\tilde\alpha)\Big],\quad \Delta_{x,j}^L \equiv \sqrt{n_x}\Big[\tilde{Q}_{U_x}^I\big(u_{x,j}^L(\tilde\alpha)\big) - u_{x,j}^L(\tilde\alpha)\Big],$$

$$D_{y,j}^H \equiv \sqrt{n_y}\big[u_{y,j}^H(\tilde\alpha) - \tau_j\big],\quad D_{x,j}^L \equiv \sqrt{n_x}\big[u_{x,j}^L(\tilde\alpha) - \tau_j\big].$$

Similarly,

$$T_{l,2}(\tilde\alpha) = T_{L,2,1} - T_{L,2,2} + O\big(n^{-1}[\log(n)]^{3/2}\big),$$

$$T_{L,2,2} \equiv \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\Big\{Q_Y'(\tau_j)\Big[\tilde{Q}_{U_y}^I\big(u_{y,j}^L(\tilde\alpha)\big) - \tau_j\Big] - Q_X'(\tau_j)\Big[\tilde{Q}_{U_x}^I\big(u_{x,j}^H(\tilde\alpha)\big) - \tau_j\Big]\Big\} < 0\right)$$

$$= \mathrm{P}\left(\sum_{j=1}^{J}\psi_j\big\{\delta Q_Y'(\tau_j)\big(\Delta_{y,j}^L + D_{y,j}^L\big) - Q_X'(\tau_j)\big(\Delta_{x,j}^H + D_{x,j}^H\big)\big\} < 0\right),$$

$$T_{L,2,1} \equiv \mathrm{P}\Bigg(\sum_{j=1}^{J}\psi_j\big\{\delta Q_Y'(\tau_j)\big(\Delta_{y,j}^L + D_{y,j}^L\big) - Q_X'(\tau_j)\big(\Delta_{x,j}^H + D_{x,j}^H\big)\big\}$$

$$< -\frac{n_x^{-1/2}}{2}\sum_{j=1}^{J}\psi_j\Big[\delta^2 Q_Y''(\tau_j)\big(\Delta_{y,j}^L + D_{y,j}^L\big)^2 - Q_X''(\tau_j)\big(\Delta_{x,j}^H + D_{x,j}^H\big)^2\Big]\Bigg),$$

$$\Delta_{y,j}^L \equiv \sqrt{n_y}\Big[\tilde{Q}_{U_y}^I\big(u_{y,j}^L(\tilde\alpha)\big) - u_{y,j}^L(\tilde\alpha)\Big],\quad \Delta_{x,j}^H \equiv \sqrt{n_x}\Big[\tilde{Q}_{U_x}^I\big(u_{x,j}^H(\tilde\alpha)\big) - u_{x,j}^H(\tilde\alpha)\Big],$$

$$D_{y,j}^L \equiv \sqrt{n_y}\big[u_{y,j}^L(\tilde\alpha) - \tau_j\big],\quad D_{x,j}^H \equiv \sqrt{n_x}\big[u_{x,j}^H(\tilde\alpha) - \tau_j\big].$$

Also, adding x and y subscripts to (52), let

$$D_{0,y}^H \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_{y,j}^H,\quad D_{0,y}^L \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_{y,j}^L = -D_{0,y}^H + O(n^{-1/2}),$$
$$D_{0,x}^L \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_{x,j}^L,\quad D_{0,x}^H \equiv \sum_{j=1}^{J}\psi_j\gamma_j D_{x,j}^H = -D_{0,x}^L + O(n^{-1/2}). \tag{92}$$

Using (47) and (48), the probability that any $\Delta_j^2 > 2\log(n)$ ("Δ_j" including $\Delta_{y,j}^H$, $\Delta_{x,j}^H$, $\Delta_{y,j}^L$, and $\Delta_{x,j}^L$) is again $O\big(n^{-2}[\log(n)]^{1/2}\big)$, much smaller than the order of magnitude in the statement of this lemma, so we can again focus on the case where all $|\Delta_j| < \sqrt{2\log(n)}$.

For the one-sided result, the remaining bounds and arguments are identical to the proof of Lemma 8.

For the two-sided result, the strategy is the same, but there are a few differences in the details. Consider $T_{H,2,2}$ first. Before, the strategy was to write the probability in terms of a quadratic function of Δ_1^H, conditional on the other Δ_j^H values. Let $\psi_1 = 1$ again, and let $\gamma_{y,j} = Q_Y'(\tau_j)/Q_Y'(\tau_1)$ and $\gamma_{x,j} = Q_X'(\tau_j)/Q_Y'(\tau_1)$, so $\gamma_{y,1} = 1$. Adding a y subscript to (60), let

$$\boldsymbol{\Delta}_{y,-1}^H \equiv \big(\Delta_{y,2}^H, \ldots, \Delta_{y,J}^H\big)',\quad \boldsymbol{\Delta}_{y,-1}^L \equiv \big(\Delta_{y,2}^L, \ldots, \Delta_{y,J}^L\big)'. \tag{93}$$

Analogous to (61), using (92), define the function $\pi_y^{H,2}(\cdot,\cdot):\mathbb{R}^{J-1}\times\mathbb{R}^J\mapsto\mathbb{R}$ as

$$\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) = D_{0,x}^L - D_{0,y}^H - \sum_{j=2}^J \psi_j\gamma_{y,j}v_j + \sum_{j=1}^J \psi_j\gamma_{x,j}w_j \tag{94}$$

for any arguments $\mathbf{v}_{-1}=(v_2,\ldots,v_J)'\in\mathbb{R}^{J-1}$ and $\mathbf{w}=(w_1,\ldots,w_J)'\in\mathbb{R}^J$. Thus, analogous to (62),

$$T_{H,2,2} = \mathrm{P}\bigl(\Delta_{y,1}^H > \delta^{-1}\pi_y^{H,2}(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L)\bigr). \tag{95}$$

Also,

$$\begin{aligned}
T_{H,2,1} &= \mathrm{P}\bigl(a(\Delta_{y,1}^H)^2 + b\Delta_{y,1}^H + c > 0\bigr),\\
&\quad a \equiv n_x^{-1/2}a_0,\quad a_0 \equiv \frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)},\\
&\quad b \equiv \delta + n_x^{-1/2}b_0,\quad b_0 \equiv D_{y,1}^H\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)},\quad b^{-1} = \delta^{-1} - n_x^{-1/2}b_0\delta^{-2} + O\bigl(n_x^{-1}\bigr),\\
&\quad c \equiv -\pi_y^{H,2}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr) + \frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}(D_{y,1}^H)^2 + \frac{n_x^{-1/2}}{2}\sum_{j=2}^J \psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\bigl(\Delta_{y,j}^H + D_{y,j}^H\bigr)^2\\
&\qquad - \frac{n_x^{-1/2}}{2}\sum_{j=1}^J \psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\bigl(\Delta_{x,j}^L + D_{x,j}^L\bigr)^2\\
&\quad = O\Bigl([\log(n)]^{1/2}\Bigr).
\end{aligned}$$

From the same arguments as before, the important root of the quadratic is

$$\begin{aligned}
r_+ &= \frac{-b + b - 2ac/b - 2a^2c^2/b^3 + O\bigl(n^{-3/2}[\log(n)]^{3/2}\bigr)}{2a} = -\frac{c}{b} - \frac{ac^2}{b^3} + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr)\\
&= -c\Bigl(\delta^{-1} - n_x^{-1/2}b_0\delta^{-2}\Bigr) - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\bigl[\pi_y^{H,2}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\bigr]^2\delta^{-3} + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr)
\end{aligned}$$

Similar to (94), define the function $\pi_y^{H,1}(\cdot,\cdot):\mathbb{R}^{J-1}\times\mathbb{R}^J\mapsto\mathbb{R}$ so that

$$\begin{aligned}
r_+ &= \pi_y^{H,1}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\\
&= -c\Bigl(\delta^{-1} - n_x^{-1/2}b_0\delta^{-2}\Bigr) - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\bigl[\pi_y^{H,2}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\bigr]^2\delta^{-3}\\
&\quad + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr),
\end{aligned}$$

where c implicitly depends on the arguments, too. Similar to before, the other root's impact is smaller-order: $a_0 = O(1)$ by A2, so r_- is of order at least $n^{1/2}$, and the corresponding tail probability is exponentially small, a loose bound for which is $O\bigl(e^{-0.99n}\bigr)$. Altogether,

$$\begin{aligned}
T_{H,2,1} &= \mathrm{P}\bigl(\Delta_{y,1}^H > \pi_y^{H,1}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\bigr) + O\bigl(e^{-0.99n}\bigr),\\
T_{h,2} &= \mathrm{P}\bigl(\Delta_{y,1}^H > \pi_y^{H,1}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\bigr) - \mathrm{P}\bigl(\Delta_{y,1}^H > \pi_y^{H,2}\bigl(\boldsymbol{\Delta}_{y,-1}^H,\boldsymbol{\Delta}_x^L\bigr)\bigr) + O\Bigl(n^{-1}[\log(n)]^{3/2}\Bigr),
\end{aligned}$$

and

$$
\begin{aligned}
&\delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) - \pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w})\\
&= \delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) + \delta^{-1}c - n_x^{-1/2}\delta^{-2}cb_0 + n_x^{-1/2}\delta^{-1}\frac{Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\big]^2 + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= \delta^{-1}\frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big(D_{y,1}^H\big)^2 + \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\big(v_j + D_{y,j}^H\big)^2\\
&\quad - \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=1}^{J}\psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\big(w_j + D_{x,j}^L\big)^2\\
&\quad - n_x^{-1/2}\delta^{-2}\overbrace{D_{y,1}^H\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)}}^{b_0}\overbrace{\Big[-\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) + O\Big(n_x^{-1/2}[\log(n)]\Big)\Big]}^{c}\\
&\quad + n_x^{-1/2}\delta^{-1}\frac{Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\big]^2\\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big)
\end{aligned}
$$

when all $|v_j| < [\log(n)]^{1/2}$ and $|w_j| < [\log(n)]^{1/2}$.

The above arguments can be repeated for $T_{l,2}$. First, the analog of (94) is

$$
\pi_y^{L,2}(\mathbf{v}_{-1},\mathbf{w}) = D_{0,x}^H - D_{0,y}^L - \sum_{j=2}^{J}\psi_j\gamma_{y,j}v_j + \sum_{j=1}^{J}\psi_j\gamma_{x,j}w_j. \tag{96}
$$

Second,

$$
\begin{aligned}
&T_{L,2,1} = \mathrm{P}\big(a(\Delta_{y,1}^L)^2 + b\Delta_{y,1}^L + c < 0\big),\\
&\quad a \equiv n_x^{-1/2}a_0, \quad a_0 \equiv \frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)},\\
&\quad b \equiv \delta + n_x^{-1/2}b_0, \quad b_0 \equiv D_{y,1}^L\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)}, \quad b^{-1} = \delta^{-1} - n_x^{-1/2}b_0\delta^{-2} + O\big(n_x^{-1}\big),\\
&\quad c \equiv -\pi_y^{L,2}\big(\boldsymbol{\Delta}_{y,-1}^L,\boldsymbol{\Delta}_x^H\big) + \frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big(D_{y,1}^L\big)^2 + \frac{n_x^{-1/2}}{2}\sum_{j=2}^{J}\psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\big(\Delta_{y,j}^L + D_{y,j}^L\big)^2\\
&\qquad - \frac{n_x^{-1/2}}{2}\sum_{j=1}^{J}\psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\big(\Delta_{x,j}^H + D_{x,j}^H\big)^2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
T_{L,2,1} &= \mathrm{P}\big(\Delta_{y,1}^L < \pi_y^{L,1}\big(\boldsymbol{\Delta}_{y,-1}^L,\boldsymbol{\Delta}_x^H\big)\big) + O\big(e^{-0.99n}\big),\\
\pi_y^{L,1}\big(\boldsymbol{\Delta}_{y,-1}^L,\boldsymbol{\Delta}_x^H\big) &= -c\Big(\delta^{-1} - n_x^{-1/2}b_0\delta^{-2}\Big) - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[\pi_y^{L,2}\big(\boldsymbol{\Delta}_{y,-1}^L,\boldsymbol{\Delta}_x^H\big)\big]^2\delta^{-3}\\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big),
\end{aligned}
$$

where c also (implicitly) depends on the argument. Third, using (92), (94), and (96), analogous to (72) and (71),

$$
\begin{aligned}
&\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w}) = -\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) + O\Big(n^{-1/2}\Big),\\
&\pi_y^{L,1}(-\mathbf{v}_{-1},-\mathbf{w}) - \delta^{-1}\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w})\\
&= -\delta^{-1}\frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big(D_{y,1}^L\big)^2 - \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=2}^J \psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\big(-v_j + D_{y,j}^L\big)^2\\
&\quad + \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=1}^J \psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\big(-w_j + D_{x,j}^H\big)^2\\
&\quad + n_x^{-1/2}\delta^{-2}D_{y,1}^L\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)}\Big[-\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w}) + O(n^{-1/2})\Big]\\
&\quad - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w})\big]^2\delta^{-3} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= -\delta^{-1}\frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big(-D_{y,1}^H\big)^2 - \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=2}^J \psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\big(-v_j - D_{y,j}^H\big)^2\\
&\quad + \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=1}^J \psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\big(-w_j - D_{x,j}^L\big)^2\\
&\quad + n_x^{-1/2}\delta^{-2}(-D_{y,1}^H)\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\\
&\quad - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[-\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\big]^2\delta^{-3} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= -\delta^{-1}\frac{n_x^{-1/2}\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big(D_{y,1}^H\big)^2 - \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=2}^J \psi_j\delta^2\frac{Q_Y''(\tau_j)}{Q_Y'(\tau_1)}\big(v_j + D_{y,j}^H\big)^2\\
&\quad + \delta^{-1}\frac{n_x^{-1/2}}{2}\sum_{j=1}^J \psi_j\frac{Q_X''(\tau_j)}{Q_Y'(\tau_1)}\big(w_j + D_{x,j}^L\big)^2\\
&\quad - n_x^{-1/2}\delta^{-2}D_{y,1}^H\frac{\delta^2 Q_Y''(\tau_1)}{Q_Y'(\tau_1)}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\\
&\quad - n_x^{-1/2}\frac{\delta^2 Q_Y''(\tau_1)}{2Q_Y'(\tau_1)}\big[\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\big]^2\delta^{-3} + O\Big(n^{-1}[\log(n)]^{3/2}\Big)\\
&= \pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w}) - \delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) + O\Big(n^{-1}[\log(n)]^{3/2}\Big).
\end{aligned}
$$

The final obstacle is the approximation of the joint PDF of $\big(\boldsymbol{\Delta}_y^H,\boldsymbol{\Delta}_x^L\big)$ and of $\big(\boldsymbol{\Delta}_y^L,\boldsymbol{\Delta}_x^H\big)$. Since

$\mathbf{\Delta}_y^H \perp\!\!\!\perp \mathbf{\Delta}_x^L$ and $\mathbf{\Delta}_y^L \perp\!\!\!\perp \mathbf{\Delta}_x^H$ by A1, using the PDF approximations in (49),

$$
\begin{aligned}
f_{(\mathbf{\Delta}_y^H,\mathbf{\Delta}_x^L)}(\mathbf{v},\mathbf{w}) &= \overbrace{\phi_{\underline{\mathcal{V}}}(\mathbf{v})\Big[1+O\Big(n^{-1/2}[\log(n)]^2\Big)\Big]}^{\text{from (49)}}\overbrace{\phi_{\underline{\mathcal{V}}}(\mathbf{w})\Big[1+O\Big(n^{-1/2}[\log(n)]^2\Big)\Big]}^{\text{from (49)}} \\
&= \phi_{\underline{\mathcal{V}},\underline{\mathcal{V}}}(\mathbf{v},\mathbf{w})\Big[1+O\Big(n^{-1/2}[\log(n)]^2\Big)\Big], \\
f_{(\mathbf{\Delta}_y^L,\mathbf{\Delta}_x^H)}(\mathbf{v},\mathbf{w}) &= \phi_{\underline{\mathcal{V}},\underline{\mathcal{V}}}(\mathbf{v},\mathbf{w})\Big[1+O\Big(n^{-1/2}[\log(n)]^2\Big)\Big],
\end{aligned}
$$

with $\phi_{\underline{\mathcal{V}},\underline{\mathcal{V}}}$ indicating the PDF of a mean-zero normal distribution with block diagonal covariance matrix with blocks $\underline{\mathcal{V}}$ and (again) $\underline{\mathcal{V}}$.

Now, the same arguments as before apply. To be explicit, following the derivation of (69) with analogous notation yields the parallel result

$$
\begin{aligned}
T_{h,2} &= \underset{\substack{v_j^2\le 2\log(n),j\ge 2\\ w_j^2\le 2\log(n)}}{\int\cdots\int}\int_{\pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w})}^{\delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})} f_{\Delta_{y,1}^H|\mathbf{\Delta}_{y,-1}^H,\mathbf{\Delta}_x^L}(v_1\mid\mathbf{v}_{-1},\mathbf{w})f_{\mathbf{\Delta}_{y,-1}^H,\mathbf{\Delta}_x^L}(\mathbf{v}_{-1},\mathbf{w}) \\
&\qquad\qquad \times dv_1\,dv_2\ldots dv_J\,dw_1\ldots dw_J \\
&\quad + O\Big(n^{-1}[\log(n)]^{3/2}\Big) \\
&= \underset{\substack{v_j^2\le 2\log(n),j\ge 2\\ w_j^2\le 2\log(n)}}{\int\cdots\int}\big[\delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})-\pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w})\big]\phi_{V_1|\mathbf{V}_{-1},\mathbf{W}}\big(\delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\mid\mathbf{v}_{-1},\mathbf{w}\big) \\
&\qquad \times \phi_{\underline{\mathcal{V}}_{-1},\underline{\mathcal{V}}}(\mathbf{v}_{-1},\mathbf{w})\,dv_2\ldots dv_J\,dw_1\ldots dw_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big).
\end{aligned}
$$

Then, following the same steps as in the derivation of (73),

$$
\begin{aligned}
T_{l,2} &= \underset{\substack{v_j^2\le 2\log(n),j\ge 2\\ w_j^2\le 2\log(n)}}{\int\cdots\int}\big[\pi_y^{L,1}(-\mathbf{v}_{-1},-\mathbf{w})-\delta^{-1}\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w})\big] \\
&\qquad \times \phi_{V_1|\mathbf{V}_{-1},\mathbf{W}}\big(\pi_y^{L,2}(-\mathbf{v}_{-1},-\mathbf{w})\mid -\mathbf{v}_{-1},-\mathbf{w}\big)\phi_{\underline{\mathcal{V}}_{-1},\underline{\mathcal{V}}}(\mathbf{v}_{-1},\mathbf{w}) \\
&\qquad \times dv_2\ldots dv_J\,dw_1\ldots dw_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big) \\
&= \underset{\substack{v_j^2\le 2\log(n),j\ge 2\\ w_j^2\le 2\log(n)}}{\int\cdots\int}\Big[\pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w})-\delta^{-1}\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})+O\Big(n^{-1}[\log(n)]^{3/2}\Big)\Big] \\
&\qquad \times \phi_{V_1|\mathbf{V}_{-1},\mathbf{W}}\big(-\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w})\mid -\mathbf{v}_{-1},-\mathbf{w}\big)\phi_{\underline{\mathcal{V}}_{-1},\underline{\mathcal{V}}}(\mathbf{v}_{-1},\mathbf{w}) \\
&\qquad \times dv_2\ldots dv_J\,dw_1\ldots dw_J \\
&\quad + O\big(n^{-1}[\log(n)]^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{\substack{v_j^2\le 2\log(n), j\ge 2\\ w_j^2\le 2\log(n)}}\cdots\int \big[\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) - \delta^{-1}\pi_y^{H,1}(\mathbf{v}_{-1},\mathbf{w})\big]\phi_{V_1\mid\mathbf{V}_{-1},\mathbf{W}}\big(\pi_y^{H,2}(\mathbf{v}_{-1},\mathbf{w}) \mid \mathbf{v}_{-1},\mathbf{w}\big)\\
&\qquad\qquad \times \phi_{\underline{\mathcal{V}}_{-1},\underline{\mathcal{V}}}(\mathbf{v}_{-1},\mathbf{w})dv_2\dots dv_J\,dw_1\dots dw_J\\
&\quad + O\big(n^{-1}[\log(n)]^2\big)\\
&= -T_{h,2} + O\big(n^{-1}[\log(n)]^2\big),
\end{aligned}
$$

so $T_{l,2} + T_{h,2} = O\big(n^{-1}[\log(n)]^2\big)$. □

A.4.2 CPE from two-sample nuisance parameter estimation error: $E_{h,2}$, $E_{l,2}$

Lemma 12. *Under the assumptions of Theorem 10, the term $E_{h,2}$ from (91) is of order $E_{h,2} = O\big(m^{-1}\log(n) + (m/n)^2\big)$, and similarly $E_{l,2} = O\big(m^{-1}\log(n) + (m/n)^2\big)$ for the corresponding upper one-sided term, where m is the common rate of smoothing parameters, $m_j \asymp m$ for all j.*

Proof. The proof largely parallels that of Lemma 9; here, we walk through the differences. Notation remains the same, but with subscripts x and y referring to the two samples and populations.

Notational modifications aside, the first difference is in the decomposition of $E_{h,2}$:

$$
\begin{aligned}
E_{h,2} &= \mathrm{P}\left(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H} > \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big]\right)\\
&\quad - \mathrm{P}\left(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,y}^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,x}^H} > \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big]\right)\\
&= E_{h,2}^1 + E_{h,2}^2,\\
E_{h,2}^1 &\equiv \mathbb{E}\bigg\{\mathrm{P}\Big(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H}\\
&\qquad\qquad < \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big] \mid \hat{\boldsymbol{\gamma}}\Big)\\
&\qquad - \mathrm{P}\Big(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H}\\
&\qquad\qquad\quad < \delta\sqrt{n_y}\boldsymbol{\psi}'\Big[Q_Y(\boldsymbol{\tau}) - Q_Y(\hat{\mathbf{u}}_y^H)\Big] - \sqrt{n_x}\boldsymbol{\psi}'\Big[Q_X(\boldsymbol{\tau}) - Q_X(\hat{\mathbf{u}}_x^H)\Big] \mid \hat{\boldsymbol{\gamma}}\Big)\bigg\},\\
E_{h,2}^2 &\equiv \mathbb{E}\bigg\{\mathrm{P}\Big(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,y}^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\mathbf{u}_{0,x}^H}\\
&\qquad\qquad < \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big] \mid \hat{\boldsymbol{\gamma}}\Big)\\
&\qquad - \mathrm{P}\Big(\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H}\\
&\qquad\qquad\quad < \delta\sqrt{n_y}\boldsymbol{\psi}'\big[Q_Y(\boldsymbol{\tau}) - Q_Y(\mathbf{u}_{0,y}^H)\big] - \sqrt{n_x}\boldsymbol{\psi}'\big[Q_X(\boldsymbol{\tau}) - Q_X(\mathbf{u}_{0,x}^H)\big] \mid \hat{\boldsymbol{\gamma}}\Big)\bigg\}.
\end{aligned}
$$

In the one-sample proof, there was a single $\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}^H}$, whose PDF is normal up to a multiplicative approximation error. Here, we have such a random variable for each sample, and we need the PDF of the difference $\delta\mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_y^H} - \mathbb{W}_{\mathbf{C},\boldsymbol{\Lambda}}^{\hat{\mathbf{u}}_x^H}$. By A1, the samples are drawn independently, so the random variables are independent, and the PDF of the difference can be derived via convolution. Heuristically (to simplify notation), if random variable W has PDF $\phi_{\sigma_W^2}(w)[1 + O(r_n)]$ and Z has PDF

$\phi_{\sigma_Z^2}(z)[1+O(r_n)]$, with $W \perp\!\!\!\perp Z$, then $\delta W - Z$ has PDF

$$\begin{aligned} f_{\delta W - Z}(t) &= \int_{\mathbb{R}} f_{W,Z}(w, \delta w - t)\, dw \\ &= \int_{\mathbb{R}} \overbrace{f_W(w) f_Z(\delta w - t)}^{\text{since } W \perp\!\!\!\perp Z}\, dw \\ &= \int_{\mathbb{R}} \phi_{\sigma_W^2}(w)[1+O(r_n)]\phi_{\sigma_Z^2}(\delta w - t)[1+O(r_n)]\, dw \\ &= \overbrace{\int_{\mathbb{R}} \phi_{\sigma_W^2}(w)\phi_{\sigma_Z^2}(\delta w - t)\, dw}^{\text{difference of normals has normal PDF}}\, [1+O(r_n)] \\ &= \phi_{\delta^2\sigma_W^2+\sigma_Z^2}(t)[1+O(r_n)]. \end{aligned}$$

So again we have a normal PDF, up to a multiplicative error of the same order as before. Adding x and y subscripts to the notation from the proof of Lemma 9, the final variance is

$$\mathcal{V}^{0,H} \equiv \delta^2 \mathcal{V}^{0,H}_{\boldsymbol{\psi},y} + \mathcal{V}^{0,H}_{\boldsymbol{\psi},x} + O\Big(m^{-1/2}n^{-1/2}\log(n) + mn^{-3/2}\Big). \tag{97}$$

The same arguments as in the proof of Lemma 9 now apply. Following the derivation of (83), given any values of $\hat{\mathbf{u}}_x^H$ and $\hat{\mathbf{u}}_y^H$,

$$\begin{aligned} E^1_{h,2}\Big(\hat{\mathbf{u}}_x^H, \hat{\mathbf{u}}_y^H\Big) &= \sqrt{n}_x \boldsymbol{\psi}'\Big\{\Big[Q_Y(\hat{\mathbf{u}}_y^H) - Q_Y(\mathbf{u}_{0,y}^H)\Big] - \Big[Q_X(\hat{\mathbf{u}}_x^H) - Q_X(\mathbf{u}_{0,x}^H)\Big]\Big\}\phi_{\mathcal{V}^{0,H}}(\tilde{w}) \\ &\quad \times \Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big] \\ &= \boldsymbol{\psi}'\Big\{\delta\Big[\widehat{Q'_Y}(\boldsymbol{\tau}) - Q'_Y(\boldsymbol{\tau})\Big] - \Big[\widehat{Q'_X}(\boldsymbol{\tau}) - Q'_X(\boldsymbol{\tau})\Big]\Big\}\phi_{\mathcal{V}^{0,H}}(\tilde{w}) \\ &\quad \times \Big[1 + O\Big(n^{-1/2}[\log(n)]^3\Big)\Big], \end{aligned}$$

where $\tilde{w}$ (from the mean value theorem) is again $O(1)$. Then, mirroring the derivation of (84),

$$\begin{aligned} E^1_{h,2} &= O(\mathbb{E}(AB)), & A &= 1 + O\Big(m^{-1/2}\log(n) + mn^{-1}\Big), \\ B &= \max_j B_j, & B_j &= \delta\,\overbrace{\Big[\widehat{Q'_Y}(\tau_j) - Q'_Y(\tau_j)\Big]}^{B_y} - \overbrace{\Big[\widehat{Q'_X}(\tau_j) - Q'_X(\tau_j)\Big]}^{B_x}. \end{aligned}$$

By linearity of the expectation operator,

$$\mathbb{E}(\delta B_y - B_x) = \delta\,\mathbb{E}(B_y) - \mathbb{E}(B_x) = O\big(m^2/n^2\big)$$

like before. Since $B_x \perp\!\!\!\perp B_y$, $\mathrm{Var}(\delta B_y - B_x) = \delta^2\,\mathrm{Var}(B_y) + \mathrm{Var}(B_x) = O\big(m^{-1}\big)$ like before, too. The remainder of the arguments about $E^1_{h,2}$ are identical to those for E^1_h in the proof of Lemma 9.

For $E^2_{h,2}$, other than notational changes, the arguments are identical to those for E^2_h in the proof of Lemma 9, with the covariance matrix changing from $\mathcal{V}^{\mathbf{t}}_{\boldsymbol{\psi}}$ to $\delta^2\mathcal{V}^{\mathbf{t}_y}_{\boldsymbol{\psi},y} + \mathcal{V}^{\mathbf{t}_x}_{\boldsymbol{\psi},x}$. □

A.5 Proof of Theorem 5

Proof. This proof shares the same structure as that of GK Theorem 6, with four main components needed. To establish the order of the CPE term due to applying the unconditional method to the local sample (i.e., CPE_{U}), first, it must be shown that $N_n \asymp nh^d$ almost surely, and second, A2 must be satisfied uniformly by the "local PDF" from which the local sample is drawn (which changes with n). Third, the order of the CPE due to bias (i.e., CPE_{Bias}) is needed. Fourth, the sum $\text{CPE}_{\text{U}} + \text{CPE}_{\text{Bias}}$ can be minimized to derive the CPE-optimal bandwidth rate and corresponding CPE. Steps two and three can be taken directly from the proof of GK Theorem 6; we comment on them but refer to the other paper for details. As in Chaudhuri (1991), we consider a deterministic bandwidth sequence, leaving treatment of a random (data-dependent) bandwidth to future work.

First, although it is random, the local sample size N_n is almost surely of order nh^d (exactly, not just $O(nh^d)$), as shown in the proof of GK Theorem 6, following the argument in Chaudhuri (1991, proof of Thm. 3.1, p. 769). Specifically, using Bernstein's Inequality and the Borel–Cantelli Lemma, it can be shown that there exist constants c_1 and c_2 such that $c_1 nh^d \le N_n \le c_2 nh^d$ for large enough n with probability one. For joint or CIQR inference, because the same bandwidth is used at each quantile, there is a single local sample and single N_n, so the prior result in Chaudhuri (1991) applies directly. For CQD inference, there are two local samples, so some additional arguments are required. For the $T_i = 0$ subsample, define the event $A_{n0} \equiv \{c_{01} nh_0^d \le N_{n0} \le c_{02} nh_0^d\}$, and similarly let $A_{n1} \equiv \{c_{11} nh_1^d \le N_{n1} \le c_{12} nh_1^d\}$. Let $A_n \equiv A_{n0} \cap A_{n1}$. We want to show that with probability one, A_n occurs for all n larger than some value n_0, i.e., $\text{P}(\liminf A_n) = 1$. The Borel–Cantelli Lemma gives this conclusion if $\sum_{n=1}^{\infty}[1 - \text{P}(A_n)] < \infty$. Using probability/set identities and inequalities, using notation A^{c} for the complement of event A,

$$\begin{aligned}
\text{P}(A_{n0} \cap A_{n1}) &= 1 - \overbrace{\text{P}(A_{n0}^{\mathsf{c}} \cup A_{n1}^{\mathsf{c}})}^{\le \text{P}(A_{n0}^{\mathsf{c}}) + \text{P}(A_{n1}^{\mathsf{c}})} \ge 1 - \text{P}(A_{n0}^{\mathsf{c}}) - \text{P}(A_{n1}^{\mathsf{c}}) = 1 - [1 - \text{P}(A_{n0})] - [1 - \text{P}(A_{n1})] \\
&= \text{P}(A_{n0}) + \text{P}(A_{n1}) - 1.
\end{aligned} \tag{98}$$

The probabilities in the RHS of (98) are bounded by the application of Bernstein's Inequality in Chaudhuri (1991). The specific constants involved will change since $\text{P}(T_i = 1)$ now enters the binomial probability parameter, but since $\text{P}(T_i = 1)$ is fixed and strictly between zero and one (from A3), the rates are the same. So, there exist constants $c_{03}, c_{04}, c_{13}, c_{14} > 0$ such that for all n,

$$\begin{aligned}
\text{P}(A_{n0}) \ge 1 - c_{03} \exp\bigl(-c_{04} nh_0^d\bigr), \quad \text{P}(A_{n1}) \ge 1 - c_{13} \exp\bigl(-c_{14} nh_1^d\bigr), \\
1 - \text{P}(A_{n0}) \le c_{03} \exp\bigl(-c_{04} nh_0^d\bigr), \quad 1 - \text{P}(A_{n1}) \le c_{13} \exp\bigl(-c_{14} nh_1^d\bigr),
\end{aligned} \tag{99}$$

Altogether,

$$\begin{aligned}
\sum_{n=1}^{\infty}[1 - \text{P}(A_n)] &= \sum_{n=1}^{\infty}[1 - \overbrace{\text{P}(A_{n0} \cap A_{n1})}^{\text{use (98)}}] \\
&\le \sum_{n=1}^{\infty}\bigl\{1 - [\text{P}(A_{n0}) + \text{P}(A_{n1}) - 1]\bigr\} \\
&= \sum_{n=1}^{\infty}\bigl\{\overbrace{1 - \text{P}(A_{n0})}^{\text{use (99)}} + \overbrace{1 - \text{P}(A_{n1})}^{\text{use (99)}}\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} c_{03} \exp\Big(-c_{04} n h_0^d\Big) + \sum_{n=1}^{\infty} c_{13} \exp\Big(-c_{14} n h_1^d\Big) \\
&= c_{03} \sum_{n=1}^{\infty} \exp(-c_{04} \overbrace{n h_0^d}^{\gtrsim [\log(n)]^2 \text{ by A8}}) + c_{13} \sum_{n=1}^{\infty} \exp(-c_{14} \overbrace{n h_1^d}^{\gtrsim [\log(n)]^2 \text{ by A8}}) \\
&\le c_{03} \sum_{n=1}^{\infty} \exp\big\{-c_{04} [\log(n)]^2\big\} + c_{13} \sum_{n=1}^{\infty} \exp\big\{-c_{14} [\log(n)]^2\big\},
\end{aligned}
$$

and both sums are finite by comparison with

$$
\sum_{n=1}^{\infty} \exp\{-2\log(n)\} = \sum_{n=1}^{\infty} \exp\big\{\log\big(n^{-2}\big)\big\} = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6.
$$

This means the summability condition from the Borel–Cantelli Lemma is satisfied, so as desired $\mathrm{P}(\liminf A_n) = 1$, and $N_{n0} \asymp n h_0^d$ and $N_{n1} \asymp n h_1^d$, where h_0 and h_1 are the same rate by assumption.

Second, in addition to having N_n instead of n, having a local distribution that changes with n is another difference with the unconditional setting. Specifically, these local PDFs must uniformly satisfy A2 for large enough n. This is shown to be true in the proof of GK Theorem 6, by using $h \to 0$ (and thus $C_h \to \{\mathbf{x}_0\}$) from A8 along with the assumed smoothness from A4–A7. For CQD inference, since the same assumptions hold conditional on $T = 0$ and $T = 1$ alike, the same argument applies. Consequently, the CPE due to application of the unconditional method to the local sample ($\mathrm{CPE}_{\mathrm{U}}$ in the main text) is obtained by replacing n with nh^d in the unconditional results. For example, two-sided QD or IQR CIs have unconditional CPE of order $O\big(n^{-2/3}\log(n)\big)$; for CQD and CIQR, replacing n with $N_n \asymp nh^d$ leaves $O\big((nh^d)^{-2/3}\log(nh^d)\big)$, where $h_0, h_1 \asymp h$ is the common bandwidth rate for the CQD case.

Third, the other component of overall CPE is from bias. In the proof of GK Theorem 6, this is $O(N_n^{1/2}h^2) = O(n^{1/2}h^{2+d/2})$: the bias is $O(h^2)$, the CI endpoint PDF is proportional to $N_n^{1/2}$, and (using the MVT) their product gives the order of CPE due to bias. For two-sided inference on a single conditional quantile, GK show that some cancellation occurs to reduce the order of magnitude, but this does not seem to occur for CQD or CIQR inference. If such cancellation did occur, then CPE would be even better (smaller) than in the results given here.

Fourth, we derive the CPE-optimal bandwidth rates and optimal CPE rates for all conditional methods. For joint inference on multiple conditional quantiles, whether one-sided or two-sided, the CPE from Theorem 2 is $O(N_n^{-1})$, so setting $\mathrm{CPE}_{\mathrm{U}} = \mathrm{CPE}_{\mathrm{Bias}}$ gives

$$
N_n^{-1} \asymp N_n^{1/2} h^2 \implies (nh^d)^{3/2} \asymp h^{-2} \implies h^* \asymp n^{-3/(4+3d)},
$$

and the overall CPE is

$$
O\big([n(h^*)^d]^{-1}\big) = O\big((n^{1-3d/(4+3d)})^{-1}\big) = O\big(n^{-4/(4+3d)}\big).
$$

For one-sided CIQR or CQD inference (or more general linear combinations or differences thereof), the CPE from Theorem 7 or Theorem 10 is $O\big(N_n^{-1/2}\log(N_n)\big)$. Ignoring the $\log(N_n)$ for simplicity,

$$
N_n^{-1/2} \asymp N_n^{1/2} h^2 \implies (nh^d)^{-1} \asymp h^2 \implies h^* \asymp n^{-1/(2+d)},
$$

and the overall CPE is $O\big([n(h^*)^d]^{-1/2}\log\big(n(h^*)^d\big)\big) = O\big(n^{-1/(2+d)}\log(n)\big)$. For two-sided CIQR or CQD inference (or the more general versions), the CPE from Theorem 7 or Theorem 10 is $O\big(N_n^{-2/3}\log(N_n)\big)$. Ignoring the $\log(N_n)$ for simplicity,

$$N_n^{-2/3} = N_n^{1/2}h^2 \implies h^* \asymp n^{-7/(12+7d)}, \text{CPE} = O\Big(n^{-8/(12+7d)}\log(n)\Big). \qquad \square$$

B Nuisance parameter estimation and plug-in bandwidth details

B.1 Nuisance parameter estimation

Selection of $\tilde{\alpha}$ in Sections 3.2 and 3.3 requires preliminary estimation of derivatives of the quantile function. We recommend the "quantile spacing" estimator first proposed by Siddiqui (1960), given earlier in (12). In practice, results are often very similar when using fractional order statistics in (12) instead of rounding to integers, or even using a kernel density estimator instead, but we do not explore those here. Below we derive a rule for bandwidth selection that ensures an optimal order of CPE, but our results are also not particularly sensitive to the bandwidth choice.

Suppressing the j subscript for simplicity, the smoothing parameter rate $m \asymp n^{2/3}$ gives the most accurate CIs (up to $\log(n)$ terms) because the orders of E_h and E_l are $O\big(m^{-1}\log(n) + m^2/n^2\big)$, where m^{-1} is the order of the variance of $\widehat{Q'}(\tau_j)$ and m^2/n^2 is the order of its bias. Ideally, we could derive more precise expressions of E_h and E_l that could then be minimized over m; for now, we just consider rates.

From (2.5) and (2.6) in Bloch and Gastwirth (1968), up to smaller-order terms,

$$\text{Var}\Big(\widehat{Q'}(\tau)\Big) \doteq m^{-1}\frac{[Q'(\tau)]^2}{2}, \quad \text{Bias}\Big(\widehat{Q'}(\tau)\Big) \doteq (m/n)^2\frac{Q'''(\tau)}{6}.$$

The derivatives of a quantile function $Q(\cdot) = F^{-1}(\cdot)$ are (mostly just applying the chain rule for derivatives)

$$Q'(\tau) = \frac{1}{f(Q(\tau))} = [f(Q(\tau))]^{-1}, \tag{100}$$

$$Q''(\tau) = -[f(Q(\tau))]^{-2}f'(Q(\tau))Q'(\tau) = -\frac{1}{[f(Q(\tau))]^2}\frac{f'(Q(\tau))}{f(Q(\tau))} = -\frac{f'(Q(\tau))}{[f(Q(\tau))]^3}, \tag{101}$$

$$\begin{aligned} Q'''(\tau) &= -\frac{f''(Q(\tau))}{f(Q(\tau))}\frac{1}{[f(Q(\tau))]^3} + \big[-f'(Q(\tau))\big]\left[-3\frac{1}{[f(Q(\tau))]^4}\frac{f'(Q(\tau))}{f(Q(\tau))}\right] \\ &= \frac{-f''(Q(\tau))f(Q(\tau)) + 3[f'(Q(\tau))]^2}{[f(Q(\tau))]^5}. \end{aligned} \tag{102}$$

For a normal distribution with mean μ and variance σ^2, the PDF can be written in terms of the standard normal PDF $\phi(\cdot)$, as $\sigma^{-1}\phi([x-\mu]/\sigma)$, and the quantile function as

$$Q(\tau) = \mu + \sigma\Phi^{-1}(\tau), \tag{103}$$

so

$$f(Q(\tau)) = \sigma^{-1}\phi\left(\frac{\mu + \sigma\Phi^{-1}(\tau) - \mu}{\sigma}\right) = \sigma^{-1}\phi\big(\Phi^{-1}(\tau)\big). \tag{104}$$

Taking a derivative,

$$f'(x) = \sigma^{-1}\phi'([x-\mu]/\sigma)\sigma^{-1} = \sigma^{-2}\frac{-[x-\mu]}{\sigma}\phi([x-\mu]/\sigma) = -\frac{x-\mu}{\sigma^2}f(x), \tag{105}$$

which can also be seen from taking a derivative of $\exp\bigl\{-(x-\mu)^2/(2\sigma^2)\bigr\}$ in x. For the second derivative, using the product rule,

$$f''(x) = -\sigma^{-2}f(x) - \frac{x-\mu}{\sigma^2}f'(x) = f(x)\left[-\sigma^{-2} + \frac{(x-\mu)^2}{\sigma^4}\right] = \sigma^{-2}\left[\left(\frac{x-\mu}{\sigma}\right)^2 - 1\right]f(x). \tag{106}$$

Substituting (103)–(106) into (102), for a $\mathrm{N}\bigl(\mu,\sigma^2\bigr)$ distribution,

$$\begin{aligned}
Q'''(\tau) &= \frac{-\sigma^{-2}\left[\left(\frac{Q(\tau)-\mu}{\sigma}\right)^2 - 1\right]f(Q(\tau))f(Q(\tau)) + 3\left[-\frac{Q(\tau)-\mu}{\sigma^2}f(Q(\tau))\right]^2}{[f(Q(\tau))]^5} \\
&= \frac{-\sigma^{-2}\left[\left(\frac{Q(\tau)-\mu}{\sigma}\right)^2 - 1\right] + 3\sigma^{-2}\left(\frac{Q(\tau)-\mu}{\sigma}\right)^2}{[f(Q(\tau))]^3} \\
&= \frac{\sigma^{-2} - \sigma^{-2}\left(\frac{Q(\tau)-\mu}{\sigma}\right)^2 + 3\sigma^{-2}\left(\frac{Q(\tau)-\mu}{\sigma}\right)^2}{[\sigma^{-1}\phi(\Phi^{-1}(\tau))]^3} \\
&= \frac{\sigma^{-2} + 2\sigma^{-2}\bigl[\Phi^{-1}(\tau)\bigr]^2}{\sigma^{-3}[\phi(\Phi^{-1}(\tau))]^3} \\
&= \frac{1 + 2\bigl[\Phi^{-1}(\tau)\bigr]^2}{\sigma^{-1}[\phi(\Phi^{-1}(\tau))]^3}
\end{aligned}$$

One option for m that achieves the CPE-optimal rate is to minimize the sum of the bias and variance of $\widehat{Q'}(\tau)$. The FOC is

$$\begin{aligned}
0 &= \frac{\partial}{\partial m}(m/n)^2\frac{Q'''(\tau)}{6} + m^{-1}\frac{[Q'(\tau)]^2}{2} = 2m/n^2\frac{Q'''(\tau)}{6} - m^{-2}\frac{[Q'(\tau)]^2}{2}, \\
&\implies 2(m/n)^2\frac{Q'''(\tau)}{6} = m^{-1}\frac{[Q'(\tau)]^2}{2}.
\end{aligned} \tag{107}$$

Using a "Gaussian plug-in" approach like in the famous bandwidth of Silverman (1986), i.e., computing the quantile derivatives for a $\mathrm{N}\bigl(\mu,\sigma^2\bigr)$ distribution,

$$\begin{aligned}
m^3 &= n^2\frac{6[Q'(\tau)]^2}{4Q'''(\tau)} = n^2\frac{3[Q'(\tau)]^2}{2Q'''(\tau)} = n^2\frac{3\sigma^{-1}\bigl[\phi\bigl(\Phi^{-1}(\tau)\bigr)\bigr]^3}{2\Bigl\{1+2[\Phi^{-1}(\tau)]^2\Bigr\}[\sigma^{-1}\phi(\Phi^{-1}(\tau))]^2} \\
&= n^2\frac{3\sigma\phi\bigl(\Phi^{-1}(\tau)\bigr)}{2+4[\Phi^{-1}(\tau)]^2}, \\
m &= n^{2/3}\left(1.5\frac{\sigma\phi\bigl(\Phi^{-1}(\tau)\bigr)}{1+2[\Phi^{-1}(\tau)]^2}\right)^{1/3}.
\end{aligned} \tag{108}$$

Similar to the suggestion by Silverman (1986), for robustness to distributions lacking a second moment, one can take $\hat{\sigma}$ to be the interquartile range divided by 1.349 (which equals σ for a normal distribution). Note that the same $\hat{\sigma}$ is used for all m_j, while different $\tau = \tau_j$ are used in (108).

Alternatively, to remove the dependence on scale through σ, we may normalize by the true $Q'(\tau_j)$ and consider the bias and variance of $\widehat{Q'}(\tau_j)/Q'(\tau_j)$:

$$\begin{aligned}
\operatorname{Var}\Big(\widehat{Q'}(\tau)/Q'(\tau)\Big) &= \frac{\operatorname{Var}\Big(\widehat{Q'}(\tau)\Big)}{[Q'(\tau)]^2} \doteq m^{-1}\frac{[Q'(\tau)]^2}{2[Q'(\tau)]^2} = 1/(2m), \\
\operatorname{Bias}\Big(\widehat{Q'}(\tau)/Q'(\tau)\Big) &= \frac{\operatorname{Bias}\Big(\widehat{Q'}(\tau)\Big)}{Q'(\tau)} \doteq (m/n)^2\frac{Q'''(\tau)}{6Q'(\tau)}, \\
0 &= \frac{\partial}{\partial m}\left[(m/n)^2\frac{Q'''(\tau)}{6Q'(\tau)} + (1/2)m^{-1}\right] = 2m/n^2\frac{Q'''(\tau)}{6Q'(\tau)} - (1/2)m^{-2}, \\
&\implies m^3 = n^2\frac{3Q'(\tau)}{2Q'''(\tau)}.
\end{aligned}$$

This is the same as before but with one fewer $Q'(\tau)$ in the numerator, so instead of (108),

$$m = n^{2/3}\left(1.5\frac{\big[\phi\big(\Phi^{-1}(\tau)\big)\big]^2}{1+2[\Phi^{-1}(\tau)]^2}\right)^{1/3}. \tag{109}$$

This is the bandwidth we use in our code, after replacing m with m_j and τ with τ_j.

Interestingly, the expression in (109) is very similar to the Gaussian plug-in bandwidth based on (3.1) in Hall and Sheather (1988), as in (16) of Kaplan (2015):

$$m_{HS} = n^{2/3}z_{1-\alpha/2}^{2/3}\left(1.5\frac{\big[\phi\big(\Phi^{-1}(\tau)\big)\big]^2}{1+2[\Phi^{-1}(\tau)]^2}\right)^{1/3}.$$

This is also the same as the bandwidth m_K proposed by Kaplan (2015) up to a constant multiple. Their suggestions are for a different setting (i.e., inference with a single, Studentized quantile), but they provide the CPE-optimal rate from our Theorems 7 and 10 and are scale-invariant (i.e., no σ). Consequently, we suggest using

$$m_j = n^{2/3}\left(1.5\frac{\big[\phi\big(\Phi^{-1}(\tau_j)\big)\big]^2}{1+2[\Phi^{-1}(\tau_j)]^2}\right)^{1/3}. \tag{110}$$

In simulations, our overall quantile inference method was not sensitive to either the choice of bandwidth or even the choice of sparsity estimation method (spacing estimator versus inverse of kernel estimator). The results were quite good with our quantile spacing estimators and plug-in smoothing parameters, and they were extremely similar with kernel density estimators using Silverman's (1986) rule of thumb bandwidth, which minimizes the kernel density estimator's (asymptotic) mean squared error under normality.

B.2 Plug-in bandwidth for conditional inference

The following suggestions are all implemented in the code available on the latter author's website.

Since analytic expressions for unconditional CPE do not exist for the methods considered here, we recommend multiplying the single quantile plug-in bandwidth in GK Section 4.3 by the appropriate power of n to achieve the optimal rate from Theorem 5. Note that the bandwidth values in GK are only for $d = 1$; only rates are given for $d > 1$.

The one-sided rate for joint inference over multiple quantiles is the same as for a single quantile. For simplicity, we suggest a common bandwidth for all quantiles, using $\tau = \arg\min_{\tau_j} \tau_j(1-\tau_j)$ in the single quantile plug-in bandwidth, and we suggest using α instead of $\tilde{\alpha}$ in the plug-in bandwidth formula; neither choice affects the asymptotic bandwidth rate. For two-sided joint inference, we recommend further multiplying the bandwidth by $n^{-2/[(2+d)(4+3d)]}$ to get the optimal rate.

For one-sided inference on linear combinations of quantiles, we recommend multiplying the single quantile plug-in bandwidth (in GK Section 4.3) by n to the power of $8/[(12 + 7d)(4 + 3d)]$ to get a bandwidth with the optimal rate. This time $\tilde{\alpha} \geq \alpha$, so we suggest plugging in the calibrated $\tilde{\alpha}$ that would be used if the sample size were n rather than N_n, and again whichever τ_j minimizes $\tau_j(1 - \tau_j)$. For two-sided inference on linear combinations, we similarly recommend multiplying the single quantile plug-in bandwidth by $n^{-2/[(12+7d)(2+d)]}$ to get a bandwidth with the optimal rate.

For quantile differences, the adjustment is the same as for linear combinations, but with separate bandwidths for the $T = 0$ and $T = 1$ samples. We again recommend using $\tau = \arg\min_{\tau_j} \tau_j(1 - \tau_j)$, and for the one-sided case, the $\tilde{\alpha}$ that would result from sample size n.

C Implementation of methods

In Supplemental Appendices C.1–C.4, we provide detailed steps for implementing our methods. For practical use, computer code implementing all methods is available from the latter author's website. The remaining subsections in this section discuss nuisance parameter estimation and our plug-in bandwidth; these are also implemented in the available code.

C.1 Steps for unconditional joint inference

To construct a $100(1 - \alpha)\%$ confidence set for $Q(\boldsymbol{\tau}) = (Q(\tau_1), \ldots, Q(\tau_J))$:

1. Parameters: determine the sample size n, the J quantile indices of interest $\tau_j \in (0, 1)$, and the desired coverage level $1 - \alpha$.

2. Calibration of $\tilde{\alpha}$: using a numerical solver, plus simulated random variables from a beta distribution or numeric integration, solve for $\tilde{\alpha}$ in (10). Let $B_j^h \equiv \tilde{Q}_U^I[u_j^h(\tilde{\alpha})]$ and $B_j^l \equiv \tilde{Q}_U^I[u_j^l(\tilde{\alpha})]$, where $u_j^h(\tilde{\alpha})$ and $u_j^l(\tilde{\alpha})$ are as defined in (27). To explain the simulation step, define $\tilde{\mathbf{u}}$ as a $2J \times 1$ vector containing all the elements $u_j^h(\tilde{\alpha})$, $u_j^l(\tilde{\alpha})$ for $j \in \{1, \ldots, J\}$ sorted in ascending order, and let $\tilde{\mathbf{B}} = \tilde{Q}_U^I[\tilde{\mathbf{u}}]$ be the corresponding vector containing elements of type B_j^l and B_j^h. We simulate draws of $\tilde{\mathbf{B}}$ according to

$$\tilde{B}_j = \tilde{B}_{j-1} + (1 - \tilde{B}_{j-1})\Delta_j, \quad \Delta_j \sim \text{Beta}((n+1)(\tilde{u}_j - \tilde{u}_{j-1}), (n+1)(1 - \tilde{u}_j)), \quad j = 1, \ldots, 2J,$$

   where $\tilde{B}_0 = 0$, $\tilde{u}_0 = 0$, and the Δ_j are all independent; any alternative method of generating random Dirichlet draws (e.g., with Gamma random variables) is also fine.

   For any given $\tilde{\alpha}$, many (e.g., 10^5 or 10^6) random samples can be drawn,[25] and the probability

[25] Let $1 - \hat{\alpha}(\tilde{\alpha})$ denote the simulated overall CP given $\tilde{\alpha}$, T the search tolerance such that the search for $\tilde{\alpha}$ stops when $|\hat{\alpha}(\tilde{\alpha}) - \alpha| < T$. With M random draws, for the $\tilde{\alpha}$ such that the true $\alpha(\tilde{\alpha}) = \alpha + K$, and assuming M is large enough to approximate the binomial distribution with the following normal,

$$\hat{\alpha}(\tilde{\alpha}) \sim \text{N}(\alpha + K, (\alpha + K)(1 - \alpha - K)/M).$$

on the RHS of (10) is the proportion of samples in which $\left(\cap_{j=1}^{J}\left[B_j^h > \tau_j\right]\right) \cap \left(\cap_{j=1}^{J}\left[B_j^l < \tau_j\right]\right)$. The calibrated $\tilde{\alpha}$ is the value that solves (10), which can be found by numerical search.

3. CI construction: individual $1-\tilde{\alpha}$ CIs are constructed for each $Q(\tau_j)$ as in Section 3 of Goldman and Kaplan (2017), i.e., by solving (8) numerically for the k_j^h and k_j^l (given $\alpha = \tilde{\alpha}$) and then plugging the solutions into (5) to compute the CI endpoints. The Cartesian product of the individual CIs is the overall $1-\alpha$ confidence set for the vector $Q(\boldsymbol{\tau})$.

C.2 Steps for unconditional linear combination inference

To construct one-sample $100(1-\alpha)\%$ CIs for linear combinations of quantiles:

1. Parameters: determine the sample size n, the J quantile indices of interest $\tau_j \in (0,1)$, and the desired coverage level $1-\alpha$.

2. Nuisance parameter estimation: using the method in Supplemental Appendix B.1, estimate $Q'(\tau_j)$ for all $j = 1, \ldots, J$; i.e., using the estimator in (12) with the formula for m_j in (110). Note: in simulations, using a standard kernel density estimator (evaluated at estimated quantiles $\hat{Q}_X^L(\tau_j)$) with MSE-optimal bandwidth has also worked well.

3. Calibration of $\tilde{\alpha}$: using a numerical solver, plus simulated random variables from a beta distribution or numeric integration, solve for $\tilde{\alpha}$ in (36) for a lower one-sided CI or (37) for an upper one-sided CI. A two-sided CI is the intersection of upper and lower one-sided $1-\alpha/2$ CIs. To simulate (36), for example, ψ_j, τ_j, and $\widehat{Q'}(\tau_j)$ are all known values, and let $B_j \equiv \tilde{Q}_U^I[u_j^H(\tilde{\alpha})]$, where the $u_j^H(\tilde{\alpha})$ are in ascending order (by j) and defined as in (34) (adjusting for negative ψ_j), which by reference to (27) gives u_j^H as an implicit function of $\tilde{\alpha}$. As in Appendix C.1, let $\tilde{\mathbf{u}}$ contain all the $u_j^H(\tilde{\alpha})$ values sorted into ascending order, and let $\tilde{\mathbf{B}}$ be the corresponding vector of $\tilde{Q}_U^I[u_j^H(\tilde{\alpha})]$ random variables. Then for $j = 1, \ldots, J$, draws of $\tilde{B}_j$ may be simulated by

$$\tilde{B}_j = \tilde{B}_{j-1} + (1-\tilde{B}_{j-1})\Delta_j, \quad \Delta_j \sim \text{Beta}\big((n+1)(\tilde{u}_j^H(\tilde{\alpha}) - \tilde{u}_{j-1}^H(\tilde{\alpha})), (n+1)(1-\tilde{u}_j^H(\tilde{\alpha}))\big),$$

   where again $\tilde{B}_0 = 0$, $\tilde{u}_0 = 0$, and the Δ_j are all independent, and again any alternative method of generating Dirichlet draws is fine. For any $\tilde{\alpha}$ considered, many (e.g., 10^5 or 10^6) random samples can be drawn.[25] The probability on the RHS of (36) is estimated by the proportion of samples in which $\sum_{j=1}^{J} \psi_j \widehat{Q'}(\tau_j)(B_j - \tau_j) > 0$, and then $\tilde{\alpha}$ may be found by numerical search.

4. CI construction: individual $1-\tilde{\alpha}$ CIs are constructed for each $Q(\tau_j)$ as in Section 3 of Goldman and Kaplan (2017) or Step 3 in Appendix C.1. The overall $1-\alpha$ CI for the linear combination contains all values of the form $\boldsymbol{\psi}'\mathbf{q}$ with $\mathbf{q}$ in the Cartesian product of the individual CIs. For a lower one-sided CI, the jth individual CI is lower one-sided if $\psi_j > 0$ and upper one-sided otherwise; for an upper one-sided CI, the opposite is true. The overall two-sided CI is the intersection of the overall lower and upper one-sided CIs.

The probability of the numerical search not accepting this $\tilde{\alpha}$ is at least

$$\mathrm{P}(\hat{\alpha} > \alpha + T) = \mathrm{P}(\hat{\alpha} - \alpha - K > T - K) = 1 - \Phi\left(\frac{\sqrt{M}(T-K)}{\sqrt{(\alpha+K)(1-\alpha-K)}}\right).$$

For example, one could let $M = 10^5$ and solve for T such that this probability is 95% for $K = 0.001$.

C.3 Steps for unconditional QD inference

To construct two-sample $100(1-\alpha)\%$ CIs for differences of linear combinations of quantiles:

1. Parameters: determine the sample sizes n_x and n_y, the J quantile indices of interest $\tau_j \in (0,1)$, and the desired coverage level $1-\alpha$.

2. Nuisance parameter estimation: using the method in Supplemental Appendix B.1, estimate $Q'_X(\tau_j)$ and $Q'_Y(\tau_j)$ for all $j = 1, \ldots, J$; i.e., using the estimator in (12) with the formula for m_j in (110). Note: in simulations, using a standard kernel density estimator (evaluated at estimated quantiles $\hat{Q}^L_X(\tau_j)$) with MSE-optimal bandwidth has also worked well.

3. Calibration of $\tilde{\alpha}$: using a numerical solver, plus simulated random variables from a beta distribution or numeric integration, solve for $\tilde{\alpha}$ in (87) for a lower one-sided CI or (89) for an upper one-sided CI. Simulation can proceed as in Appendix C.2, with the independence of the two samples allowing us to separately draw realizations from the two Dirichlet distributions. Similar to Step 2 of Appendix C.1 or Step 3 of Appendix C.2, the $\tilde{\mathbf{u}}^H_x$, $\tilde{\mathbf{u}}^L_y$, $\tilde{\mathbf{B}}_x$, and $\tilde{\mathbf{B}}_y$ contain the values of, respectively, $\mathbf{u}^H_x$, $\mathbf{u}^L_y$, $\mathbf{B}_x$, and $\mathbf{B}_y$ sorted in ascending order. In the case of lower one-sided CI we draw

$$\begin{aligned}
&\tilde{B}_{x,j} = \tilde{B}_{x,j-1} + (1-\tilde{B}_{x,j-1})\Delta_{x,j},\\
&\Delta_{x,j} \sim \text{Beta}\big((n_x+1)(\tilde{u}^H_{x,j}(\tilde{\alpha}) - \tilde{u}^H_{x,j-1}(\tilde{\alpha})), (n_x+1)(1-\tilde{u}^H_{x,j}(\tilde{\alpha}))\big),\\
&\tilde{B}_{y,j} = \tilde{B}_{y,j-1} + (1-\tilde{B}_{y,j-1})\Delta_{y,j},\\
&\Delta_{y,j} \sim \text{Beta}\big((n_y+1)(\tilde{u}^L_{y,j}(\tilde{\alpha}) - \tilde{u}^L_{y,j-1}(\tilde{\alpha})), (n_y+1)(1-\tilde{u}^L_{y,j}(\tilde{\alpha}))\big)
\end{aligned}$$

   where $\tilde{B}_{x,0} = \tilde{B}_{y,0} = 0$, $\tilde{u}^H_{x,0} = \tilde{u}^L_{y,0} = 0$, and the Δ are all independent; alternative methods of generating Dirichlet draws are also fine. Drawing many samples allows us to calculate the RHS of (87) as the proportion of samples in which

$$\sum_{j=1}^{J} \psi_j \left[\widehat{Q'_Y}(\tau_j)(B_{y,j}-\tau_j) - \widehat{Q'_X}(\tau_j)(B_{x,j}-\tau_j)\right] > 0,$$

   which is implicitly a function of $\tilde{\alpha}$. Then, the calibrated value of $\tilde{\alpha}$ that solves (87) may be found by numerical search. For $J=1$, the normal approximation discussed in Appendix D also yields an approximate solution to $\tilde{\alpha}$. Setting $\hat{\theta} \equiv (1+\hat{\gamma}/\mu)/\sqrt{1+(\hat{\gamma}/\mu)^2}$, $\tilde{\alpha} = \Phi(z_\alpha/\hat{\theta})$ for one-sided CIs or $\tilde{\alpha}/2 = \Phi(z_{\alpha/2}/\hat{\theta})$ for two-sided; $\mu \equiv \sqrt{n_y/n_x}$, and $\hat{\gamma}$ is the estimator of $f_X(Q_X(\tau))/f_Y(Q_Y(\tau))$.

4. CI construction: individual $1-\tilde{\alpha}$ CIs are constructed for each $Q_X(\tau_j)$ and $Q_Y(\tau_j)$ as in Section 3 of Goldman and Kaplan (2017) or Step 3 in Appendix C.1. The overall $1-\alpha$ CI for the linear combination is given by (88) or (90) or the intersection of the two. For a lower one-sided CI, the individual CI for $Q_Y(\tau_j)$ is lower one-sided if $\psi_j > 0$ and upper one-sided otherwise, while the individual CI for $Q_X(\tau_j)$ is upper one-sided if $\psi_j > 0$ and lower one-sided otherwise; for an upper one-sided CI, the opposite is true. The overall two-sided CI is the intersection of the overall lower and upper one-sided $1-\alpha/2$ CIs.

C.4 Steps for conditional inference

To construct $100(1-\alpha)\%$ CIs for conditional versions of the objects of interest in Supplemental Appendices C.1–C.3:

1. Discrete covariates (if applicable): for the discrete components of $\mathbf{X}$, restrict the sample to observations where the discrete components of $\mathbf{X}_i$ equal those of $\mathbf{x}_0$, the point of interest. In the following, treat this subsample as the full sample, and treat $\mathbf{X}$ as only having the remaining continuous components.

2. Bandwidth: let $p = \min_j \tau_j$, and compute the plug-in bandwidth h_{GK} from Goldman and Kaplan (2017, §4.3, p. 336). For the one-sided plug-in bandwidth, a value of α is required; for the confidence set, this can simply be the overall α, but otherwise we suggest using the $\tilde{\alpha}$ that would be computed if $N_n = n$ (using the unconditional steps above). To compute the final h, multiply h_{GK} by the following adjustment, where d is the number of (continuous) components in $\mathbf{X}$: $n^{-2/[(2+d)(4+3d)]}$ for two-sided joint inference; $n^{8/[(12+7d)(4+3d)]}$ for a one-sided linear combination CI; $n^{-2/[(12+7d)(2+d)]}$ for a two-sided linear combination CI; and none for one-sided joint inference. For a conditional quantile difference, do this for the $T_i = 0$ subsample to get h_0 and for the $T_i = 1$ subsample to get h_1, using the same adjustments as for linear combination CIs. With $d > 1$, there is no plug-in h_{GK}, but one could a) normalize all components of $\mathbf{X}$ to have the same variance, b) compute h_{GK} separately for each component and pick the median value, c) multiply the value from (b) by the following adjustment: $n^{3/(2+d)}$ for a two-sided CI, or $n^{7/(4+3d)}$ for a one-sided CI.

3. Local sample: collect the values $\{Y_i : \|\mathbf{X}_i - \mathbf{x}_0\|_\infty \le h\}$. For a conditional quantile difference, do this separately for the subsamples with $T_i = 0$ and $T_i = 1$, using respective bandwidths h_0 and h_1.

4. CI construction: using the local sample(s), follow the steps for the corresponding unconditional CI from Supplemental Appendices C.1–C.3.

D Further approximation and intuition: two-sample QD

We now provide a more detailed version of the discussion at the end of Section 3.3. In the two-sample QD case with $J = 1$, which is similar to the Behrens–Fisher problem, we explore further approximations that have computational benefits and theoretical insights. The upper one-sided example is used for clarity.

Consider calibrating $\tilde{\alpha}$ by approximating the two independent beta random variables with normals. Let $B_x \equiv \text{Beta}\big(u_x^h(n_x+1), (1-u_x^h)(n_x+1)\big)$ and $B_y \equiv \text{Beta}\big(u_y^l(n_y+1), (1-u_y^l)(n_y+1)\big)$. By Theorem 1, convolution, $B_x \perp\!\!\!\perp B_y$, Lemma 6, and Assumption A1, and omitting smaller-order remainder terms from interpolation, estimation of γ, and local linearization of the distribution, the CP in (89) is

$$\begin{aligned} \mathrm{P}(B_x - \tau < \gamma[B_y - \tau]) &= \mathrm{P}\Big([B_x - u_x^h] - \gamma[B_y - u_y^l] < [\tau - u_x^h] + \gamma[u_y^l - \tau]\Big) \\ &= \Phi\left(\frac{[\tau - u_x^h] + \gamma[u_y^l - \tau]}{\sqrt{\gamma^2 u_y^l(1-u_y^l)/n_y + u_x^h(1-u_x^h)/n_x}}\right) + O(n^{-1/2}\log(n)) \end{aligned}$$

$$= \Phi\left(z_{1-\tilde{\alpha}} \frac{1 + (\gamma/\delta)}{\sqrt{\frac{u_x^h(1-u_x^h)}{\tau(1-\tau)} + (\gamma/\delta)^2 \frac{u_y^l(1-u_y^l)}{\tau(1-\tau)}}} \right) + O(n^{-1/2}\log(n))$$

$$= \Phi\left(z_{1-\tilde{\alpha}} \frac{1 + (\gamma/\delta)}{\sqrt{1 + (\gamma/\delta)^2}} \right) + O(n^{-1/2}\log(n)).$$

Under exchangeability (which implies $\gamma = 1$) and equal sample sizes ($\delta = 1$), $\tilde{\alpha} = \Phi(z_\alpha/\sqrt{2})$, the biggest possible value.

The calibration equation turns out to be identical for the lower one-sided case. Consequently, using $\alpha/2$ for the two one-sided cases yields the two-sided

$$\tilde{\alpha}/2 = \Phi(z_{\alpha/2}/\theta^*), \quad \theta^* \equiv \frac{1 + \gamma/\delta}{\sqrt{1 + (\gamma/\delta)^2}}.$$

E Additional simulations

E.1 Unconditional simulations

Tables 6 and 7 show additional unconditional results.

Table 6: Mean length of 0.1-quantile difference CIs, $1 - \alpha = 0.95$, $F_X = F_Y$ shown in column headers, $n_x = n_y = 25$.

	$N(0,1)$	Logistic(0,1)	Unif(0,1)	Exp(1)	$\mathrm{LogN}(0,1)$
L-stat	2.58	5.71	0.38	0.45	0.64
Normal	2.13	4.68[a]	0.45	0.77	1.02
Kaplan (2015)	2.96	6.44	0.43	0.51	0.73
Bootstrap	2.54	5.53	0.38	0.44	0.63

[a] With 2000 replications. With 10 000 replications, one CI has "infinite" (to the computer's precision) length, so the mean length is infinity.

E.2 Conditional simulations

Tables 8 and 9 are (together) an expanded version of Table 3 from the main text, Tables 11 and 12 an expansion of Table 4, and Table 13 an expansion of Table 5. Table 10 is new.

Table 14 shows results for a new simulation DGP based on the empirical application. We focused on food budget share for 2-adult, 2-child households and 1-adult, 1-child households. Separately for the two subsamples, we fit a Gaussian mixture model to the data (using R package `mclust`), using a mixture of 10 Gaussian distributions for the larger subsample and seven for the smaller subsample (with no restrictions placed on the component distributions' means or covariances). These Gaussian mixture distributions were taken to be the "true" distributions/DGP for the simulation. As in the application, we used the empirical deciles (for each simulated dataset) for the nine x_0 values. For any x_0, integrating the conditional PDF from the mixture Gaussian distribution determines the "true" (for the simulation DGP) conditional quantile. We ran 1000 replications, sampling a new dataset from the Gaussian mixtures each time (with `mclust::sim`), which also determines slightly different x_0 each time. Our method (L-stat) has CP between 93.8% and 96.4% across all x_0 and

Table 7: CP and mean length of median difference CIs, $1-\alpha = 0.90$, F_X and F_Y shown in column headers, $n_x = 51$ and $n_y = 101$.

	$F_X = \mathrm{N}(0,1)$ $F_Y = \mathrm{N}(0,5)$	$F_X = \mathrm{N}(0,1)$ $F_Y = t_5$	$F_X = \mathrm{Logistic}(0,1)$ $F_Y = \mathrm{Unif}(-10,10)$
		Coverage Probability	
L-stat	0.899	0.902	0.896
Normal	0.910	0.912	0.887
Kaplan (2015)	0.911	0.919	0.902
Bootstrap	0.893	0.895	0.890
Permutation (rep)[a]	0.809	0.893	0.778
Permutation (CR13)[b]	0.810	0.819	0.815
		Mean Interval Length	
L-stat	2.14	0.73	3.36
Normal	2.24	0.77	3.35
Kaplan (2015)	2.27	0.78	3.53
Bootstrap	2.34	0.78	3.73

[a] Our replication, with kernel-based Studentization.
[b] Calculated as $1-2R$ from the one-sided, level 5% test rejection rates, R, published in Chung and Romano (2013, Table 1), noting that all distributions are symmetric.

τ. The other methods have more under-coverage: across all x_0 and τ, the range of CP for CBS is $[0.916, 0.948]$, and the CP range for QYg is $[0.887, 0.974]$. Compared to the worst L-stat CP of 93.8%, CBS had CP below 93.8% in 13/18 cases, and QYg had CP below 93.8% in 12/18 cases. The trade-off is that the L-stat CIs are always longest in this case. Of course, if one tried to make the CBS and QYg intervals long enough to attain correct coverage (like L-stat), then they may get long enough that L-stat is the shortest.

F Additional empirical results

F.1 Quantile Engel curve differences

Figure 5 contains additional analysis.

F.2 Gift exchange

To demonstrate our two-sample QTE inference, we use the experimental data from Gneezy and List (2006, Tables I and V). In short, individuals in the control group work for a certain advertised hourly wage, while individuals in the treatment group are surprised with a larger hourly wage upon arrival. The goal is to investigate "gift exchange," specifically whether the higher wages induce higher effort (as measured by productivity). The experiment is run separately for two different tasks: data entry for a library (typing in a book's author, title, etc.), and door-to-door fundraising for a non-profit. Productivity is measured for each participant, as the number of books entered in each of four 90-minute segments for the library task, and as dollars raised before/after lunch for fundraising. The sample sizes are small: 10 and 9 for control and treatment, respectively, for the library task, and 10 and 13 for control and treatment for fundraising.

The main result of Gneezy and List (2006) is that the "gift wage" treatment raises productivity

Table 8: CP for conditional median difference CIs, $g(T_i) = 1$; $1 - \alpha = 0.95$, $n_1 = n_0 = 200$. "QYg(f)" is QYg on all replications instead of just those where L-stat may be used. The number of replications where L-stat was used for normal U_i, by x_0 in ascending order, was 902, 996, 980, 945, 966, 794, 984, 966, 999, 1000; for t_3, 949, 999, 991, 988, 989, 958, 994, 995, 998, 998; for Cauchy, 986, 999, 996, 995, 993, 995, 1000, 995, 998, 1000; for χ^2_3, 990, 997, 987, 996, 999, 980, 993, 997, 1000, 997; for uniform, 893, 994, 982, 946, 973, 777, 993, 963, 1000, 1000.

		x_0 value									
	U_i	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Coverage Probability									
L-stat	N(0, 1)	0.922	0.953	0.957	0.958	0.954	0.947	0.972	0.955	0.953	0.941
QYg	N(0, 1)	0.955	0.986	0.997	0.972	1.000	0.981	1.000	0.999	0.952	0.987
CBS	N(0, 1)	0.978	0.968	0.980	0.978	0.980	0.967	0.985	0.963	0.976	0.952
QYg(f)	N(0, 1)	0.959	0.986	0.997	0.974	1.000	0.985	1.000	0.999	0.952	0.987
C91	N(0, 1)	0.989	0.991	0.992	0.999	0.986	0.997	0.988	0.994	0.978	0.980
QYu	N(0, 1)	0.947	0.982	0.994	0.962	1.000	0.984	1.000	1.000	0.929	0.992
L-stat	t_3	0.940	0.950	0.951	0.953	0.965	0.966	0.957	0.961	0.955	0.964
QYg	t_3	0.971	0.985	0.989	0.974	0.999	0.991	1.000	0.999	0.951	0.993
CBS	t_3	0.989	0.979	0.977	0.973	0.986	0.979	0.983	0.965	0.971	0.970
QYg(f)	t_3	0.972	0.985	0.989	0.974	0.999	0.991	1.000	0.999	0.951	0.993
C91	t_3	0.993	0.997	0.992	0.999	0.995	0.999	0.991	0.998	0.992	0.992
QYu	t_3	0.970	0.984	0.987	0.961	1.000	0.985	1.000	0.996	0.923	0.991
L-stat	Cauchy	0.942	0.964	0.941	0.958	0.962	0.962	0.962	0.960	0.957	0.965
QYg	Cauchy	0.989	0.990	0.971	0.990	0.985	0.993	0.975	0.998	0.974	0.982
CBS	Cauchy	0.999	0.995	0.981	0.991	0.996	0.986	0.997	0.988	0.992	0.993
QYg(f)	Cauchy	0.989	0.990	0.971	0.990	0.985	0.993	0.975	0.998	0.974	0.982
C91	Cauchy	0.998	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
QYu	Cauchy	0.985	0.986	0.963	0.987	0.977	0.992	0.955	0.994	0.954	0.985
L-stat	χ^2_3	0.951	0.958	0.935	0.956	0.958	0.948	0.952	0.958	0.951	0.965
QYg	χ^2_3	0.956	0.961	0.948	0.962	0.992	0.971	0.964	0.976	0.948	0.967
CBS	χ^2_3	0.986	0.977	0.954	0.973	0.973	0.974	0.970	0.972	0.970	0.975
QYg(f)	χ^2_3	0.956	0.961	0.949	0.962	0.992	0.972	0.964	0.976	0.948	0.967
C91	χ^2_3	0.996	0.987	0.950	0.994	0.994	0.993	0.977	0.989	0.983	0.977
QYu	χ^2_3	0.955	0.961	0.946	0.975	0.990	0.971	0.955	0.980	0.927	0.963
L-stat	Uniform	0.944	0.967	0.956	0.959	0.955	0.954	0.965	0.946	0.943	0.952
QYg	Uniform	0.957	0.988	0.995	0.975	1.000	0.987	0.999	0.992	0.901	0.966
CBS	Uniform	0.963	0.970	0.960	0.967	0.971	0.968	0.968	0.962	0.932	0.950
QYg(f)	Uniform	0.961	0.988	0.995	0.976	1.000	0.990	0.999	0.992	0.901	0.966
C91	Uniform	0.966	0.991	0.974	0.997	0.974	0.996	0.959	0.983	0.943	0.957
QYu	Uniform	0.949	0.986	0.993	0.966	1.000	0.977	0.999	0.993	0.893	0.976

Table 9: Median CI lengths, corresponding to Table 8.

		x_0 value									
	U_i	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Median Interval Length									
L-stat	$\mathrm{N}(0,1)$	0.545	0.532	0.515	0.594	0.514	0.576	0.465	0.450	0.366	0.348
QYg	$\mathrm{N}(0,1)$	0.472	0.507	0.665	0.434	0.795	0.470	0.683	0.495	0.400	0.343
CBS	$\mathrm{N}(0,1)$	0.729	0.449	0.610	0.504	0.621	0.619	0.525	0.466	0.393	0.365
QYg(f)	$\mathrm{N}(0,1)$	0.472	0.508	0.665	0.435	0.795	0.473	0.684	0.497	0.400	0.343
C91	$\mathrm{N}(0,1)$	0.719	0.592	0.723	0.812	0.621	0.857	0.523	0.651	0.399	0.422
QYu	$\mathrm{N}(0,1)$	0.476	0.512	0.671	0.438	0.802	0.475	0.690	0.500	0.404	0.346
L-stat	t_3	0.588	0.592	0.568	0.656	0.599	0.625	0.526	0.501	0.405	0.397
QYg	t_3	0.536	0.561	0.698	0.489	0.811	0.512	0.727	0.544	0.445	0.402
CBS	t_3	0.825	0.531	0.690	0.521	0.732	0.596	0.612	0.498	0.449	0.429
QYg(f)	t_3	0.536	0.561	0.698	0.490	0.811	0.513	0.727	0.544	0.445	0.401
C91	t_3	0.885	0.682	0.861	0.840	0.781	0.921	0.659	0.735	0.503	0.523
QYu	t_3	0.541	0.567	0.705	0.494	0.818	0.517	0.733	0.549	0.449	0.405
L-stat	Cauchy	0.783	0.759	0.752	0.803	0.813	0.752	0.687	0.626	0.523	0.515
QYg	Cauchy	0.945	0.804	0.927	0.774	0.957	0.818	0.895	0.837	0.696	0.660
CBS	Cauchy	1.661	0.874	1.203	0.760	1.350	0.754	0.990	0.695	0.724	0.649
QYg(f)	Cauchy	0.945	0.804	0.927	0.774	0.957	0.818	0.895	0.837	0.696	0.660
C91	Cauchy	2.586	2.127	2.302	2.137	2.396	2.052	2.234	2.110	2.021	2.014
QYu	Cauchy	0.954	0.812	0.935	0.781	0.965	0.826	0.903	0.845	0.703	0.666
L-stat	χ^2_3	0.870	0.776	0.959	0.877	0.912	0.955	0.869	0.778	0.715	0.702
QYg	χ^2_3	0.677	0.684	0.760	0.606	0.857	0.630	0.794	0.631	0.622	0.556
CBS	χ^2_3	1.222	0.894	1.145	0.890	1.009	1.051	0.987	0.823	0.768	0.769
QYg(f)	χ^2_3	0.677	0.685	0.759	0.606	0.857	0.630	0.795	0.631	0.622	0.556
C91	χ^2_3	1.233	0.898	1.085	0.996	1.089	1.181	0.889	0.926	0.783	0.758
QYu	χ^2_3	0.683	0.691	0.767	0.612	0.865	0.636	0.801	0.637	0.628	0.561
L-stat	Uniform	0.602	0.522	0.602	0.610	0.604	0.637	0.551	0.537	0.443	0.430
QYg	Uniform	0.463	0.513	0.648	0.441	0.777	0.463	0.686	0.496	0.416	0.360
CBS	Uniform	0.771	0.520	0.679	0.588	0.694	0.733	0.589	0.588	0.460	0.453
QYg(f)	Uniform	0.465	0.513	0.649	0.442	0.779	0.465	0.686	0.496	0.416	0.360
C91	Uniform	0.738	0.595	0.738	0.805	0.672	0.866	0.558	0.681	0.441	0.456
QYu	Uniform	0.467	0.518	0.654	0.445	0.784	0.467	0.692	0.501	0.420	0.363

Table 10: CP and median length for CQD CIs with normal U_i, $g(T_i) = 1$; $1 - \alpha = 0.95$, $n_1 = n_0 = 200$. "QYg(f)" is QYg on all replications instead of just those where L-stat may be used. The number of replications where L-stat was used for $\tau = 0.25$, by x_0 in ascending order, was 902, 911, 837, 786, 939, 775, 926, 920, 997, 994; for $\tau = 0.75$, 558, 940, 893, 654, 778, 547, 950, 904, 987, 983.

		x_0 value									
	τ	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Coverage Probability									
L-stat	0.25	0.938	0.956	0.958	0.969	0.966	0.948	0.950	0.942	0.948	0.939
QYg	0.25	0.948	0.996	0.998	0.920	0.994	0.951	1.000	0.997	0.938	0.975
CBS	0.25	0.968	0.964	0.957	0.976	0.981	0.970	0.961	0.973	0.966	0.956
QYg(f)	0.25	0.953	0.996	0.998	0.936	0.994	0.962	1.000	0.997	0.938	0.975
C91	0.25	0.921	0.987	1.000	0.994	0.965	0.990	0.995	0.986	0.941	0.971
QYu	0.25	0.905	0.991	0.996	0.908	0.985	0.939	1.000	1.000	0.917	0.976
L-stat	0.75	0.934	0.948	0.966	0.969	0.964	0.951	0.971	0.956	0.955	0.960
QYg	0.75	0.951	0.963	0.962	0.965	1.000	0.906	0.998	0.977	0.992	0.979
CBS	0.75	0.959	0.972	0.964	0.977	0.955	0.957	0.965	0.970	0.963	0.970
QYg(f)	0.75	0.972	0.965	0.966	0.977	1.000	0.946	0.998	0.979	0.992	0.979
C91	0.75	0.996	0.989	0.947	0.992	0.991	0.986	0.946	0.988	0.984	0.979
QYu	0.75	0.938	0.963	0.942	0.958	1.000	0.923	0.995	0.971	0.989	0.986
		Median Interval Length									
L-stat	0.25	0.651	0.689	0.579	0.880	0.722	0.756	0.516	0.597	0.414	0.400
QYg	0.25	0.468	0.552	0.748	0.423	0.575	0.438	0.811	0.513	0.337	0.368
CBS	0.25	0.698	0.594	0.565	0.708	0.642	0.669	0.524	0.567	0.427	0.407
QYg(f)	0.25	0.468	0.554	0.753	0.424	0.577	0.439	0.815	0.515	0.337	0.368
C91	0.25	0.570	0.688	0.904	0.816	0.516	0.765	0.634	0.612	0.368	0.421
QYu	0.25	0.472	0.557	0.755	0.427	0.580	0.442	0.818	0.518	0.340	0.371
L-stat	0.75	0.641	0.642	0.741	0.896	0.628	0.828	0.589	0.539	0.388	0.426
QYg	0.75	0.477	0.460	0.527	0.443	0.904	0.478	0.509	0.440	0.480	0.349
CBS	0.75	0.654	0.521	0.624	0.677	0.587	0.670	0.547	0.497	0.407	0.426
QYg(f)	0.75	0.482	0.460	0.527	0.449	0.919	0.484	0.510	0.441	0.480	0.350
C91	0.75	0.872	0.602	0.553	0.816	0.791	0.754	0.456	0.593	0.475	0.432
QYu	0.75	0.481	0.464	0.532	0.447	0.912	0.482	0.514	0.444	0.484	0.352

Table 11: CP and median length for CQD CIs with normal U_i, $g(T_i) = 1 - 2T_i$; $1 - \alpha = 0.95$, $n_1 = n_0 = 200$. "QYg(f)" is QYg on all replications instead of just those where L-stat may be used. The number of replications where L-stat was used for $\tau = 0.25$, by x_0 in ascending order, was 760, 962, 909, 757, 887, 694, 970, 923, 992, 995; for $\tau = 0.50$, 901, 994, 972, 948, 973, 764, 988, 961, 1000, 1000; for $p = 0.75$, 672, 855, 856, 678, 806, 614, 936, 868, 992, 990.

		x_0 value									
	τ	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Coverage Probability									
L-stat	0.25	0.851	0.959	0.915	0.948	0.954	0.955	0.955	0.952	0.942	0.959
QYg	0.25	0.273	0.991	0.763	0.901	0.981	0.897	0.999	0.987	0.942	0.982
CBS	0.25	0.936	0.968	0.929	0.980	0.967	0.966	0.962	0.976	0.968	0.965
QYg(f)	0.25	0.342	0.991	0.779	0.923	0.983	0.926	0.999	0.988	0.942	0.982
C91	0.25	0.936	0.991	0.958	0.988	0.976	0.981	0.984	0.991	0.970	0.975
QYu	0.25	0.202	0.990	0.623	0.910	0.970	0.902	1.000	0.982	0.929	0.973
L-stat	0.50	0.895	0.965	0.926	0.953	0.951	0.952	0.954	0.957	0.940	0.946
QYg	0.50	0.321	0.990	0.885	0.929	0.998	0.952	0.999	0.999	0.928	0.987
CBS	0.50	0.976	0.970	0.977	0.966	0.980	0.962	0.977	0.969	0.962	0.957
QYg(f)	0.50	0.343	0.990	0.887	0.932	0.998	0.963	0.999	0.999	0.928	0.987
C91	0.50	0.967	0.997	0.980	0.997	0.983	0.997	0.981	0.993	0.974	0.985
QYu	0.50	0.248	0.990	0.769	0.913	0.994	0.939	0.999	0.998	0.916	0.990
L-stat	0.75	0.869	0.956	0.928	0.962	0.952	0.969	0.955	0.957	0.931	0.948
QYg	0.75	0.212	0.985	0.756	0.881	0.970	0.868	0.999	0.976	0.953	0.989
CBS	0.75	0.935	0.962	0.938	0.974	0.973	0.977	0.969	0.962	0.956	0.955
QYg(f)	0.75	0.293	0.987	0.781	0.916	0.976	0.915	0.999	0.979	0.953	0.989
C91	0.75	0.929	0.980	0.951	0.980	0.989	0.990	0.958	0.985	0.975	0.970
QYu	0.75	0.161	0.987	0.633	0.891	0.967	0.888	0.999	0.978	0.951	0.987

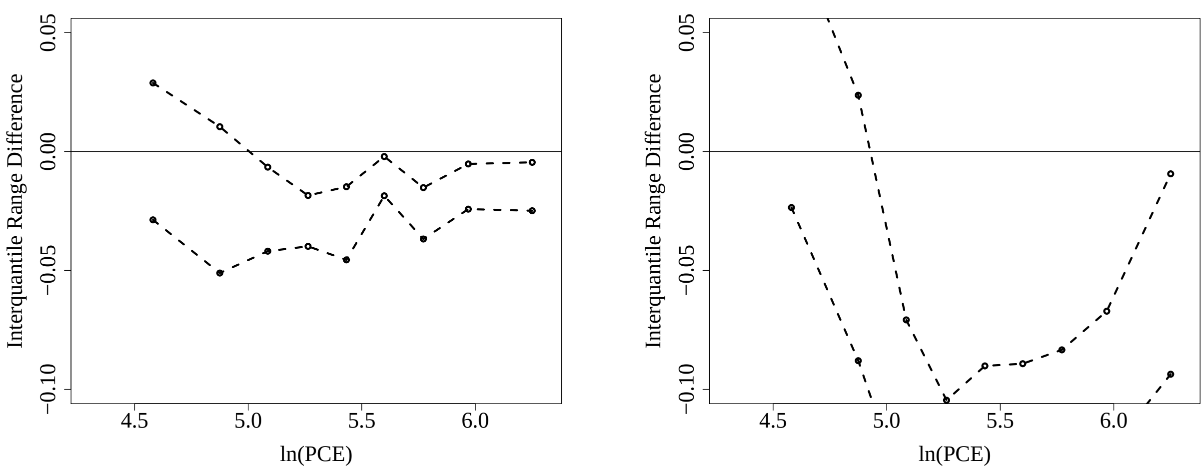

Figure 5: Pointwise 90% confidence intervals (connected for visual ease) for conditional interquantile range (0.9-quantile minus median) differences in alcohol (left) or housing and utilities (right) budget share between two-adult and one-adult childless households, conditional on real log total per capita expenditure.

Table 12: Median CI lengths corresponding to Table 11.

		x_0 value									
	τ	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Median Interval Length									
L-stat	0.25	0.638	0.638	0.607	0.823	0.660	0.760	0.526	0.576	0.407	0.405
QYg	0.25	0.493	0.573	0.727	0.436	0.871	0.462	0.767	0.517	0.458	0.354
CBS	0.25	0.693	0.558	0.597	0.696	0.611	0.687	0.526	0.550	0.419	0.413
QYg(f)	0.25	0.499	0.576	0.730	0.438	0.873	0.459	0.770	0.519	0.458	0.354
C91	0.25	0.716	0.649	0.724	0.821	0.645	0.762	0.559	0.619	0.408	0.425
QYu	0.25	0.498	0.578	0.734	0.440	0.879	0.467	0.775	0.522	0.462	0.357
L-stat	0.50	0.545	0.540	0.511	0.583	0.526	0.591	0.459	0.461	0.366	0.355
QYg	0.50	0.514	0.587	0.780	0.436	0.916	0.472	0.789	0.543	0.454	0.336
CBS	0.50	0.750	0.452	0.609	0.498	0.630	0.610	0.519	0.475	0.396	0.371
QYg(f)	0.50	0.514	0.587	0.780	0.437	0.917	0.477	0.791	0.543	0.454	0.336
C91	0.50	0.723	0.603	0.719	0.808	0.628	0.856	0.519	0.662	0.401	0.423
QYu	0.50	0.519	0.592	0.787	0.440	0.925	0.476	0.797	0.548	0.458	0.339
L-stat	0.75	0.627	0.632	0.672	0.918	0.721	0.813	0.534	0.572	0.409	0.398
QYg	0.75	0.495	0.570	0.736	0.443	0.857	0.460	0.770	0.512	0.470	0.355
CBS	0.75	0.672	0.549	0.584	0.678	0.605	0.654	0.521	0.521	0.419	0.414
QYg(f)	0.75	0.497	0.578	0.736	0.444	0.857	0.466	0.775	0.517	0.470	0.355
C91	0.75	0.711	0.622	0.723	0.798	0.664	0.771	0.514	0.600	0.435	0.421
QYu	0.75	0.500	0.576	0.743	0.447	0.864	0.465	0.777	0.517	0.474	0.358

Table 13: CP and median length for conditional IQR CIs; $1-\alpha = 0.95$, $n_1 = n_0 = 200$. "QYg(f)" is QYg on all replications instead of just those where L-stat may be used. The number of replications where L-stat was used for normal U_i, by x_0 in ascending order, was 813, 922, 872, 691, 918, 773, 962, 963, 999, 994; for t_3, 907, 953, 898, 815, 964, 868, 967, 970, 997, 996; for Cauchy, 953, 976, 910, 872, 976, 903, 964, 977, 996, 988; for χ^2_3, 959, 993, 970, 963, 988, 970, 984, 993, 999, 997.

		x_0 value									
	U_i	0.050	0.087	0.125	0.181	0.237	0.324	0.411	0.558	0.706	0.853
		Coverage Probability									
L-stat	N(0, 1)	0.980	0.964	0.979	0.974	0.981	0.966	0.977	0.953	0.968	0.966
QYg	N(0, 1)	0.906	0.919	0.989	0.352	1.000	0.656	1.000	0.952	0.978	0.963
QYg(f)	N(0, 1)	0.921	0.925	0.990	0.428	1.000	0.710	1.000	0.954	0.978	0.963
QYu	N(0, 1)	0.857	0.951	0.985	0.462	1.000	0.750	1.000	0.965	0.979	0.968
L-stat	t_3	0.986	0.935	0.981	0.966	0.975	0.964	0.968	0.952	0.979	0.970
QYg	t_3	0.892	0.917	0.967	0.441	0.998	0.709	1.000	0.950	0.972	0.952
QYg(f)	t_3	0.901	0.921	0.970	0.479	0.998	0.737	1.000	0.951	0.972	0.952
QYu	t_3	0.871	0.937	0.986	0.551	1.000	0.791	1.000	0.961	0.975	0.952
L-stat	Cauchy	0.978	0.893	0.969	0.930	0.974	0.966	0.977	0.934	0.965	0.973
QYg	Cauchy	0.940	0.887	0.921	0.804	0.951	0.892	0.934	0.954	0.920	0.935
QYg(f)	Cauchy	0.942	0.889	0.928	0.824	0.952	0.901	0.936	0.955	0.920	0.936
QYu	Cauchy	0.932	0.890	0.897	0.846	0.929	0.901	0.944	0.938	0.909	0.914
L-stat	χ^2_3	0.948	0.945	0.968	0.963	0.963	0.959	0.952	0.951	0.967	0.953
QYg	χ^2_3	0.905	0.952	0.972	0.912	0.979	0.879	0.968	0.933	0.977	0.939
QYg(f)	χ^2_3	0.909	0.952	0.973	0.915	0.979	0.882	0.969	0.933	0.977	0.939
QYu	χ^2_3	0.904	0.953	0.962	0.914	0.982	0.917	0.966	0.954	0.977	0.951
		Median Interval Length									
L-stat	N(0, 1)	0.379	0.422	0.408	0.451	0.362	0.405	0.350	0.324	0.251	0.278
QYg	N(0, 1)	0.282	0.292	0.424	0.244	0.491	0.270	0.402	0.274	0.242	0.198
QYg(f)	N(0, 1)	0.284	0.293	0.422	0.245	0.497	0.271	0.402	0.275	0.242	0.198
QYu	N(0, 1)	0.285	0.295	0.428	0.246	0.496	0.272	0.406	0.277	0.244	0.200
L-stat	t_3	0.482	0.487	0.539	0.560	0.451	0.532	0.463	0.409	0.308	0.357
QYg	t_3	0.326	0.333	0.438	0.281	0.510	0.303	0.432	0.306	0.273	0.239
QYg(f)	t_3	0.327	0.334	0.439	0.281	0.511	0.303	0.434	0.306	0.273	0.239
QYu	t_3	0.329	0.336	0.442	0.283	0.515	0.306	0.436	0.308	0.275	0.241
L-stat	Cauchy	0.902	0.856	1.245	1.158	0.786	1.088	0.968	0.789	0.599	0.647
QYg	Cauchy	0.626	0.550	0.630	0.519	0.627	0.507	0.559	0.503	0.467	0.440
QYg(f)	Cauchy	0.628	0.549	0.632	0.518	0.627	0.508	0.559	0.504	0.467	0.439
QYu	Cauchy	0.632	0.555	0.636	0.524	0.633	0.512	0.564	0.508	0.472	0.444
L-stat	χ^2_3	0.766	0.640	0.796	0.769	0.692	0.742	0.709	0.643	0.593	0.614
QYg	χ^2_3	0.467	0.457	0.543	0.412	0.610	0.401	0.502	0.406	0.445	0.390
QYg(f)	χ^2_3	0.466	0.457	0.543	0.413	0.610	0.401	0.502	0.406	0.445	0.390
QYu	χ^2_3	0.471	0.461	0.548	0.416	0.616	0.405	0.507	0.410	0.449	0.393

Table 14: CP and median length for conditional median difference CIs at empirical deciles, DGP based on Gaussian mixture model fitted to food expenditure data for 2-adult/2-child ($T = 1$) and 1-adult/1-child ($T = 0$) households; $1 - \alpha = 0.95$, $n_0 = 2490$, $n_1 = 7095$, 1000 simulation replications.

		x_0 empirical quantile								
Method	τ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					Coverage Probability					
L-stat	0.50	0.944	0.958	0.956	0.952	0.948	0.950	0.952	0.940	0.950
QYg	0.50	0.907	0.900	0.938	0.930	0.924	0.974	0.939	0.967	0.925
CBS	0.50	0.933	0.925	0.929	0.930	0.916	0.942	0.925	0.919	0.936
L-stat	0.90	0.964	0.944	0.948	0.945	0.946	0.953	0.953	0.938	0.956
QYg	0.90	0.945	0.952	0.922	0.926	0.921	0.928	0.926	0.887	0.905
CBS	0.90	0.947	0.944	0.932	0.935	0.932	0.941	0.935	0.927	0.948
					Median Interval Length					
L-stat	0.50	0.070	0.049	0.040	0.035	0.029	0.026	0.027	0.027	0.036
QYg	0.50	0.046	0.037	0.031	0.027	0.024	0.023	0.023	0.024	0.028
CBS	0.50	0.056	0.040	0.032	0.028	0.024	0.021	0.022	0.022	0.030
L-stat	0.90	0.099	0.081	0.064	0.054	0.051	0.047	0.045	0.053	0.048
QYg	0.90	0.062	0.055	0.046	0.041	0.037	0.033	0.032	0.030	0.029
CBS	0.90	0.077	0.063	0.052	0.044	0.041	0.037	0.037	0.042	0.039

significantly in the first period, but not significantly thereafter. We do not simply re-test this main result (though we indeed support it) but rather offer complementary analysis on quantile treatment effects.

For the library task, Gneezy and List (2006) performed two types of one-sided 5% tests: a Wilcoxon rank-sum (a.k.a. Mann–Whitney–Wilcoxon or Mann–Whitney U) test, and an unequal variances t-test. For the first 90-minute period, the null was (barely) rejected by each test in favor of the treatment productivity being higher. For the remaining 90-minute periods, the null was not rejected by either test. The goal of the rank-sum test is to reject if $\mathrm{P}(T > C) > 0.5$, where T is a random variable corresponding to treatment group productivity and similarly C is control group productivity. The t-test instead tests for equality of means, though the assumption of normality of the sample average productivities is questionable with such a small sample size (too small to rely on the CLT).

Complementing these original tests for the library task, our method tests for equality at a chosen quantile of the productivity distribution. Also using a one-sided 5% test, we do not reject the null of equality at the lower quartile or the median, but we do reject at the upper quartile. Our two-sided, equal-tailed, 90% CIs are given in Table 15. The results are consistent with the rank-sum result that the first period treatment productivity is higher overall in some sense, and consistent with the t-test result that the mean is higher. Our test further suggests that the shift comes primarily (though not exclusively) from the upper part of the distribution: for the library task in the first period, the gift wage seems to induce the most productive workers to become extremely productive, while the effect is much less (if any) on less productive workers. For periods 2–4 (period 2 shown in table), our test fails to reject the null at any of these quartiles, consistent with the original results.

For fundraising, Gneezy and List (2006) report one-sided 1% significance for the rank-sum test in

Table 15: Two-sided 90% confidence intervals for quartile treatment effects in Gneezy and List (2006).

Period (method)[a]	Lower quartile	Median	Upper quartile
		Library task[b]	
1 (kern/MSE)	(-10.25,21.42)	(-2.45,26.27)	(2.31,30.01)
1 (spac/bias)	(-10.45,21.67)	(-2.40,26.19)	(2.46,29.76)
2 (kern/MSE)	(-15.20,8.48)	(-7.23,25.53)	(-15.76,28.08)
2 (spac/bias)	(-16.56,8.96)	(-7.66,28.31)	(-16.05,28.25)
		Fundraising task[c]	
1 (kern/MSE)	(7.45,26.50)	(-1.45,27.11)	(-14.11,22.96)
1 (spac/bias)	(7.54,26.32)	(-0.98,27.00)	(-14.06,22.76)
2 (kern/MSE)	(-3.96,13.70)	(-5.58,11.47)	(-14.93,10.96)
2 (spac/bias)	(-3.86,13.67)	(-5.68,11.53)	(-20.13,12.04)

[a] Methods for $\hat{\gamma}$ estimation are abbreviated "kern/MSE" for a kernel density estimator with Silverman's (1986) rule of thumb bandwidth (MSE-optimal under normality), and "spac/bias" for a quantile spacing estimator with our zero-bias smoothing parameter.
[b] Units: books logged per period (90 minutes).
[c] Units: dollars raised per period (three hours).

period one.[26] We again use a 5% one-sided QTE test (10% two-sided), this time finding significance at the *lower* quartile (and almost at the median), but not at the upper quartile. The two-sided CIs are in Table 15. This is consistent with the original results, as is our failure to reject the null at any quartile in period two. Most interestingly, our results suggest that the *less* productive workers are most affected by the gift wage for the fundraising task, whereas the more productive workers were most affected in the library task.

In cases like Gneezy and List (2006), it may be even more appropriate to test for treatment effects across the entire distribution, which can be interpreted as multiple QTE testing that controls the familywise error rate (FWER). Such a method is developed in Goldman and Kaplan (2016), who find that the fundraising effect at lower quantiles is still significant with FWER controlled at 10% (two-sided), although the library effect is (barely) not.

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[26] We compute a slightly higher p-value of 0.03. They do not report a Wilcoxon result for the second period, but we compute a p-value of 0.19. They also do not report a t-test this time.

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